FYMM 3/Fall 2012 **Problem set 10**

To be returned on Wednesday, November 21, 10.15am at latest

1. Show that the wedge product $f \wedge g$ of $f \in \Omega^k(V)$ and $g \in \Omega^l(V)$ is really in $\Omega^{k+l}(V)$.

2. With the notation of (1), show that $f \wedge g = (-1)^{kl} g \wedge f$.

3. Prove that the wedge product is associative.

4. There is a coordinate independent formula for the Lie derivative of a differential form. It is given as

$$(\mathcal{L}_X \omega)(X_1, \ldots, X_k) = X \cdot \omega(X_1, X_2, \ldots, X_k) - \sum_{i=1}^{k} \omega(X_1, \ldots, [X, X_i], \ldots, X_k)$$

where the X_i's are arbitrary vector fields. The first term is just the directional derivative (in X direction) of the smooth function $\omega(X_1, \ldots, X_k)$ obtained by evaluating the differential form point-wise on the manifold. Derive from this the formula for the Lie derivative in terms of the components $\omega = \sum \omega_{i_1 \cdots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ in a local coordinate system.

5. Let X be a vector field and ω a differential form of degree k on a manifold. The interior product $i_X \omega$ is defined to be a form of degree $k-1$ given as

$$(i_X \omega)(X_1, X_2, \ldots, X_{k-1}) = \omega(X, X_1, X_2, ..., X_{k-1})$$

for vector fields $X_1, \ldots, X_{k-1}$. Prove the relation $\mathcal{L}_X = d \circ i_X + i_X \circ d$. (You can use local coordinates, but it is also possible to prove the relation in the coordinate free formalism.)

6. Use the result of exercise 5 to show that Lie derivative and exterior differentiation commute.