MARTINGALES AND HARMONIC ANALYSIS

Solutions to exercises

4. Petermichl's dyadic shift and the Hilbert transform.

4.1 Write the binary expansion $x = \operatorname{sgn}(x)\sum_{i\in\mathbb{Z}} \alpha_i 2^{-i}$, where $\alpha_i = \alpha_i(x) \in \{0,1\}$ for all $i \in \mathbb{Z}$ and $\alpha_i = 0$ for every $i < i_0$ where i_0 is the first integer with $2^{-i_0} \geq |x|$. For $x > 0$ and any $j \in \mathbb{Z}$, we may write

$$x = \begin{cases} \sum_{i>j} \alpha_i 2^{-i}, & j < i_0 \\ \sum_{i=i_0}^{j} \alpha_i 2^{-i} + \sum_{i>j} \alpha_i 2^{-i}, & j \geq i_0. \end{cases}$$

For the case $x < 0$ and any $j \in \mathbb{Z}$,

$$x = \left(\sum_{i>j} 2^{-i} - 2^{-j}\right) - \sum_{i\in\mathbb{Z}} \alpha_i 2^{-i} = \begin{cases} -2^{-j} + \sum_{i>j}(1-\alpha_i)2^{-i}, & j < i_0 \\ -2^{-j} - \sum_{i=i_0}^{j} \alpha_i 2^{-i} + \sum_{i>j}(1-\alpha_i)2^{-i}, & j \geq i_0. \end{cases}$$

Thus, for every $j \in \mathbb{Z}$, using the fact that $\mathscr{D}_j \pm 2^{-i} = \mathscr{D}_j$ for all $i \leq j$, we obtain

$$\mathscr{D}_j + x = \begin{cases} \mathscr{D}_j + \sum_{i>j} \alpha_i 2^{-i}, & x > 0 \\ \mathscr{D}_j + \sum_{i>j}(1-\alpha_i)2^{-i}, & x < 0. \end{cases}$$

It follows that $\mathscr{D} + x = \mathscr{D}^{\beta(x)}$, where $\beta_i(x) = \alpha_i(x)$ if $x > 0$ and $\beta_i(x) = 1 - \alpha_i(x)$ if $x < 0$. In both cases, the sequence $(\beta_i(x))_{i<i_0(x)}$ is a constant (in the first case, zero; in the second, one). Hence any $\mathscr{D}^\beta$, for which $\lim_{i\to-\infty} \beta_i$ does not exist, cannot be of the form $\mathscr{D} + x$ for any $x \in \mathbb{R}$.

4.2 Write (in this exercise) $\mathscr{D}_j^1$ for the one-dimensional dyadic intervals of length $2^{-j}, j \in \mathbb{Z}$. Then $\mathscr{D}_j = \mathscr{D}_j^1 \times \mathscr{D}_j^1, j \in \mathbb{Z}$, and one can define (e.g.) $\mathscr{D}_{j+1/2} := \mathscr{D}_{j+1}^1 \times \mathscr{D}_j^1, j \in \mathbb{Z}$. Then, denoting $\mathscr{F}_i = \sigma(\mathscr{D}_i), i \in \frac{1}{2}\mathbb{Z}$,

$$\begin{aligned} f &= \sum_{j=-\infty}^{\infty} \sum_{Q\in\mathscr{D}_j} 1_Q\big(\mathbb{E}[f|\mathscr{F}_{j+1}] - \mathbb{E}[f|\mathscr{F}_j]\big) \\ &= \sum_{j=-\infty}^{\infty} \sum_{Q\in\mathscr{D}_j} 1_Q\big(\mathbb{E}[f|\mathscr{F}_{j+1}] - \mathbb{E}[f|\mathscr{F}_{j+1/2}] + \mathbb{E}[f|\mathscr{F}_{j+1/2}] - \mathbb{E}[f|\mathscr{F}_j]\big) \end{aligned}$$

where

$$\mathbb{E}[f|\mathscr{F}_i] = \sum_{Q\in\mathscr{D}_i} \frac{1_Q}{|Q|}\int_Q f, \quad i \in \frac{1}{2}\mathbb{Z}.$$

For $Q = I \times J \in \mathscr{D}_j$, write $Q_s := I_s \times J$ and $Q_{st} := I_s \times J_t$ where $s,t \in \{-,+\}$. Note that $Q = Q_+ \cup Q_-$ and $|Q| = 2|Q_s|$, and further, $Q_s = Q_{s-} \cup Q_{s+}$ and $|Q_s| = 2|Q_{st}|$. Then we can write,

$$\frac{1_Q}{|Q|}\int_Q f = \sum_{s\in\{+,-\}} \frac{1_{Q_s}}{2|Q_s|}\int_{Q_s} f + \sum_{s\in\{+,-\}} \frac{1_{Q_s}}{2|Q_s|}\int_{Q_{-s}} f,$$

and similarly,

$$\frac{1_{Q_s}}{|Q_s|}\int_{Q_s} f = \sum_{t\in\{+,-\}} \frac{1_{Q_{st}}}{2|Q_{st}|}\int_{Q_{st}} f + \sum_{t\in\{+,-\}} \frac{1_{Q_{st}}}{2|Q_{st}|}\int_{Q_{s,-t}} f.$$

Version: December 14, 2012.

Thus, for $Q \in \mathscr{D}_j, j \in \mathbb{Z}$, we calculate

$$
\begin{aligned}
&1_Q\big(\mathbb{E}[f|\mathscr{F}_{j+1}] - \mathbb{E}[f|\mathscr{F}_{j+1/2}] + \mathbb{E}[f|\mathscr{F}_{j+1/2}] - \mathbb{E}[f|\mathscr{F}_j]\big) \\
&= \sum_{s\in\{+,-\}} \Big(\sum_{t\in\{+,-\}} \frac{1_{Q_{st}}}{|Q_{st}|} \int_{Q_{st}} f \Big) - \sum_{s\in\{+,-\}} \frac{1_{Q_s}}{|Q_s|} \int_{Q_s} f + \sum_{s\in\{+,-\}} \frac{1_{Q_s}}{|Q_s|} \int_{Q_s} f - \frac{1_Q}{|Q|} \int_Q f \\
&= \sum_{s\in\{+,-\}} \sum_{t\in\{+,-\}} \frac{1_{Q_{st}}}{2|Q_{st}|} \Big(\int_{Q_{st}} f - \int_{Q_{s,-t}} f \Big) + \sum_{s\in\{+,-\}} \frac{1_{Q_s}}{2|Q_s|} \Big(\int_{Q_s} f - \int_{Q_{-s}} f \Big) \\
&= \sum_{s\in\{+,-\}} \frac{1_{Q_{s-}} - 1_{Q_{s+}}}{|Q_s|} \Big(\int_{Q_{s-}} f - \int_{Q_{s+}} f \Big) + \frac{1_{Q_-} - 1_{Q_+}}{|Q|} \Big(\int_{Q_-} f - \int_{Q_+} f \Big) \\
&= \sum_{s\in\{+,-\}} h_{Q_s} \int h_{Q_s} f + h_Q \int h_Q f,
\end{aligned}
$$

where

$$
\begin{aligned}
h_{Q_s}(x,y) &:= \frac{1}{|Q_s|^{1/2}} \big(1_{Q_{s-}} - 1_{Q_{s+}}\big)(x,y) = \frac{1_{I_s}(x)}{|I_s|^{1/2}} h_J(y), \\
h_Q(x,y) &:= \frac{1}{|Q|^{1/2}} \big(1_{Q_-} - 1_{Q_+}\big)(x,y) = h_I(x) \frac{1_J(y)}{|J|^{1/2}}.
\end{aligned}
$$

4.3 As in 3.23, one can represent any $t_k \in [-1,1]$ in the form $t_k = \sum_{j=1}^\infty \varepsilon_{kj} 2^{-j}$ were $\varepsilon_{kj} \in \{-1,+1\}$. Denote

$$M := \max\{|F(t)| \colon t \in \{-1,1\}^n\}.$$

Then for any $t \in [-1,1]^n$,

$$|F(t)| = \Big| \sum_{k=1}^n t_k \xi_k \Big| = \Big| \sum_{k=1}^n \Bigg(\xi_k \sum_{j=1}^\infty \varepsilon_{kj} 2^{-j} \Bigg) \Big| \le \sum_{j=1}^\infty 2^{-j} \Big| \sum_{k=1}^n \varepsilon_{kj} \xi_k \Big| \le M \sum_{j=1}^\infty 2^{-j} = M,$$

so M is the maximum in the whole cube $[-1,1]^n$.

4.4 Without loss of generality, we may assume that $\max_{1\le k\le n} \|g_k\|_\infty = 1$. Then for a.e. fixed $x \in \mathbb{R}$, $g_k(x) \in [-1,1]$. For a moment, fix $x \in \mathbb{R}$ and for each $k \in \{1,\dots,n\}$, write $g_k(x) = \sum_{j=1}^\infty \eta_{kj} 2^{-j}$ (as we did for t_k in the previous exercise) where $\eta_{kj} = \eta_{kj}(x) \in \{-1,1\}$. Then, by the triangle inequality in the probability space,

$$
\begin{aligned}
\Bigg(\mathbb{E} \Big| \sum_{k=1}^n g_k(x) \varepsilon_k f_k(x) \Big|^p \Bigg)^{1/p} &= \Bigg(\mathbb{E} \Big| \sum_{j=1}^\infty 2^{-j} \Bigg(\sum_{k=1}^n \eta_{kj} \varepsilon_k f_k(x) \Bigg) \Big|^p \Bigg)^{1/p} \\
&\le \sum_{j=1}^\infty 2^{-j} \Bigg(\mathbb{E} \Big| \sum_{k=1}^n \eta_{kj} \varepsilon_k f_k(x) \Big|^p \Bigg)^{1/p}.
\end{aligned}
$$

Note that here, for every $j \in \mathbb{N}$, $(\eta_{kj}\varepsilon_k)_{k=1}^n$ and $(\varepsilon_k)_{k=1}^n$ have the same distribution. Thus,

$$\mathbb{E} \Big| \sum_{k=1}^n \eta_{kj} \varepsilon_k f_k(x) \Big|^p = \mathbb{E} \Big| \sum_{k=1}^n \varepsilon_k f_k(x) \Big|^p.$$

We conclude with

$$\Bigg(\mathbb{E} \Big| \sum_{k=1}^n g_k(x) \varepsilon_k f_k(x) \Big|^p \Bigg)^{1/p} \le \sum_{j=1}^\infty 2^{-j} \Bigg(\mathbb{E} \Big| \sum_{k=1}^n \varepsilon_k f_k(x) \Big|^p \Bigg)^{1/p} = \Bigg(\mathbb{E} \Big| \sum_{k=1}^n \varepsilon_k f_k(x) \Big|^p \Bigg)^{1/p}.$$

It remains to take the pth power and to integrate over $\mathbb{R}$.

5. Back to Burkholder's inequality.

5.1 For any $0 < a \le b \le 2a$ and $x \in \mathbb{R}$,

$$|H_{ab}f(x)| = \Big| \int_{a<|y|<b} \frac{f(x-y)}{\pi y}\,\mathrm{d}y \Big| \le \frac{1}{\pi a} \int_{|y|<b} |f(x-y)|\,\mathrm{d}y \le \frac{2}{\pi b} \int_{x-b}^{x+b} |f(y)|\,\mathrm{d}y \le \frac{4}{\pi} Mf(x).$$

Let $g \in C_c(\mathbb{R})$. First note that

$$\int_{a<|y|<b} \frac{g(x)}{\pi y}\,\mathrm{d}y = \frac{g(x)}{\pi} \int_{a<|y|<b} \frac{dy}{y} = 0, \quad \text{for any } x \in \mathbb{R}.$$

Thus,

$$\begin{aligned} |H_{ab}g(x)| &= \Big| \int_{a<|y|<b} \frac{1}{\pi y}[g(x-y) - g(x)]\,\mathrm{d}y \Big| \le \frac{2}{\pi b} \int_{|y|<b} |g(x-y) - g(x)|\,\mathrm{d}y \\ &\le \frac{4}{\pi} \max_{|s-t|<b} |g(s) - g(t)|, \end{aligned}$$

which tends to zero as $b \to 0$ by the continuity of g.

For a general $f \in L^p(\mathbb{R})$ and $\delta > 0$, let $g \in C_c(\mathbb{R})$ with $\|f - g\|_p < \delta$. Write $H_{ab}f = H_{ab}((f-g) + g) = H_{ab}(f-g) + H_{ab}g$. Then

$$\limsup_{b\to 0} |H_{ab}f| \le \limsup_{b\to 0} \big(|H_{ab}(f-g)| + |H_{ab}g|\big) \le \frac{4}{\pi} M(f-g).$$

Hence, for any $\varepsilon > 0$, since the maximal operator is of weak-type (p, p),

$$|\{x \in \mathbb{R}\colon \limsup_{b\to 0} |H_{ab}f(x)| > \varepsilon\}| \le |\{x \in \mathbb{R}\colon \frac{4}{\pi} M(f-g)(x) > \varepsilon\}| \le C\varepsilon^{-p} \|f - g\|_p^p \le C(\delta/\varepsilon)^p.$$

Since $\delta > 0$ can be taken arbitrarily small, this shows that $|\{x \in \mathbb{R}\colon \limsup_{b\to 0} |H_{ab}f(x)| > \varepsilon\}| = 0$ for any $\varepsilon > 0$. This proves that $\limsup_{b\to 0} |H_{ab}f| = 0$ a.e. and hence, $H_{ab}f \to 0$ a.e.