MARTINGALES AND HARMONIC ANALYSIS

Solutions to exercises

3. Martingale transforms and Burkholder's inequality.

3.2 Let $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ be a filtration and $(f_k) \subseteq L^p(\mathscr{F})$ a sequence, and abbreviate $E_k f_k := \mathbb{E}[f_k|\mathscr{F}_k]$. Observe that the sequence $(E_k f_k)$ is a martingale. Also note that, by Jensen's inequality, $(E_k f_k)^2 \le E_k f_k^2$.

Let $p \ge 2$. By duality in $L^{p/2}$,

$$\|(E_k f_k)_k\|_{L^p(\ell^2)}^2 = \left(\int \left(\sum_k (E_k f_k)^2\right)^{p/2} d\mu\right)^{2/p} = \sup\left\{\int \sum_k (E_k f_k)^2 \cdot \phi \, d\mu \colon \|\phi\|_{(p/2)'} \le 1\right\}$$

For any such ϕ,

$$\int \sum_k (E_k f_k)^2 \cdot \phi \, d\mu \le \int \sum_k E_k f_k^2 \cdot |\phi| \, d\mu = \int \sum_k f_k^2 \cdot E_k|\phi| \, d\mu \le \int M\phi \sum_k f_k^2 \, d\mu$$
$$\le \|M\phi\|_{L^{(p/2)'}} \cdot \|(f_k)\|_{L^p(\ell^2)}^2;$$

where M is the maximal function $Mf := \sup_{k\in\mathbb{Z}} |E_k f| = \|(E_k f)\|_{\ell^\infty}$. By Doob's inequality,

$$\|M\phi\|_{(p/2)'} \le p/2\|\phi\|_{(p/2)'} \le p/2.$$

The claim for $p \ge 2$ follows by taking supremum over all relevant ϕ.

Then suppose that $1 < p < 2$. By duality in $L^p(\ell^2)$,

$$\|(E_k f_k)_k\|_{L^p(\ell^2)}^2 = \sup\left\{\int \sum_k E_k f_k \cdot g_k \, d\mu \colon \|(g_k)\|_{L^{p'}(\ell^2)} \le 1\right\}.$$

For any such (g_k),

$$\int \sum_k E_k f_k \cdot g_k \, d\mu = \int \sum_k f_k \cdot E_k g_k = \int \sum_k \frac{f_k}{\|(f_j)\|_{L^p(\ell^2)}} E_k\big(\|(f_j)\|_{L^p(\ell^2)} \cdot g_k\big) \, d\mu$$
$$\le \left\|\Big(E_k\big(\|(f_j)\|_{L^p(\ell^2)} \cdot g_k\big)\Big)_k\right\|_{L^{p'}(\ell^2)} = \|(f_j)\|_{L^p(\ell^2)} \cdot \|(E_k g_k)_k\|_{L^{p'}(\ell^2)}$$
$$\le c_p \|(f_j)\|_{L^p(\ell^2)}$$

where we used duality in $L^{p'}(\ell^2)$ and the first part of the proof with $p' > 2$. The proof is now completed by taking supremum over all relevant (g_k).

3.3 Recall that $f_n = \sum_{k=0}^n d_k$ and that $\{d_k\}_{k=0}^n$ is an orthogonal set. Thus, by Pythagorean Theorem,

$$\|f_n\|_2^2 = \sum_{k=0}^n \|d_k\|_2^2 = \int \sum_{k=0}^n d_k^2 d\mu = \|S_n f\|_2^2 \;\Rightarrow\; \|S_n f\|_2 = \|f_n\|_2.$$

For the second claim, it is assumed that $\|F_n\|_p \le c_p \|S_n F\|_p$ is true for *all* martingales F and for some $1 < p < \infty$. We follow the hint and write

$$f_n^2 = \left(\sum_{k=0}^n d_k\right)^2 = \sum_{k=0}^n d_k^2 + 2\sum_{k=0}^n \left(\sum_{\ell=0}^{k-1} d_\ell\right) d_k = (S_n f)^2 + 2\sum_{k=0}^n f_{k-1} d_k,$$

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so that

$$f_n^2 - (S_n f)^2 = 2\sum_{k=0}^{n} f_{k-1} d_k =: 2F_n.$$

It is easy to check that $(f_{k-1}d_k)_{k=0}^n$ is again a martingale difference sequence and thus, F_n is a martingale. Then, by the assumption,

$$\|F_n\|_p \leq c_p \|S_n F\|_p;$$

here

$$S_n F = \left(\sum_{k=0}^{n} f_{k-1}^2 d_k^2\right)^{1/2} \leq \max_{0\leq i\leq n} |f_i| \left(\sum_{k=0}^{n} d_k^2\right)^{1/2} = \max_{0\leq i\leq n} |f_i| \cdot S_n f.$$

By Doob's inequality (for a finite martingale),

$$\Big\| \max_{0\leq i\leq n} |f_i| \Big\|_q \leq q' \cdot \max_{0\leq i\leq n} \|f_i\|_q \leq \|f_n\|_q,$$

where the last estimate follows by the martingale property and Fact 1. Thus,

$$\begin{aligned}\|f_n^2 - (S_n f)^2\|_p = 2\|F_n\|_p &\leq c_p \Big\| \max_{0\leq i\leq n} |f_i| \cdot S_n f \Big\|_p \leq c_p \Big\| \max_{0\leq i\leq n} |f_i| \Big\|_{2p} \cdot \|S_n f\|_{2p} \\ &\leq c_p (2p)' \|f_n\|_{2p} \cdot \|S_n f\|_{2p}\end{aligned}$$

where we used Cauchy-Schwarz. This gives us two estimates: (1)

$$\|S_n f\|_{2p}^2 = \|(S_n f)^2\|_p = \|f_n^2 + ((S_n f)^2 - f_n^2)\|_p \leq \|f_n\|_{2p}^2 + c_p \|f_n\|_{2p} \|S_n f\|_{2p}$$

and similarly, (2)

$$\|f_n\|_{2p}^2 \leq \|S_n f\|_{2p}^2 + c_p \|f_n\|_{2p} \|S_n f\|_{2p}.$$

By dividing the first estimate by $\|f_n\|_{2p}$ and denoting $X := \|S_n f\|_{2p}/\|f_n\|_{2p}$ we get the inequality $X^2 \leq 1 + c_p X$ which implies that

$$(X - c_p/2)^2 = X^2 - c_p X + (c_p/2)^2 \leq 1 + c_p^2/4 \leq (2\max\{1, c_p/2\})^2.$$

Thus,

$$X \leq c_p/2 + 2\max\{1, c_p/2\} \leq 3\max\{1, c_p\}.$$

This gives us the second estimate in the assertion with a constant $c_{2p} \leq 3\max\{1, c_p\}$. The first estimate is obtained by similar considerations (i.e. dividing the second estimate by $\|S_n f\|_{2p}$ and denoting $Y := \|f_n\|_{2p}/\|f_n\|_{2p}$).

3.4 Let's denote by (1) $\|S_n f\|_p \leq C_p \|f_n\|_p$ and by (2) $\frac{1}{C_p}\|f_n\|_p \leq \|S_n f\|_p$. By the previous exercise we know both inequalities are fulfilled for $p = 2^k$. First we use that $p = 2^k$ satifies (2), then applying Marcinkiewicz interpolation, we obtain (2) for all $p \in [2, \infty)$. Duality gives us (1) is satisfied for all $p \in (1, 2]$. Similarly, by previous exercise, (1) it's true for $p = 2^k$, so by duality, (2) it's satisfied for $p = (2^k)'$. Applying Marcinkiewicz interpolation we get that (2) it's true for $p \in (1, 2]$, therefore, (1) it's satisfied, by duality, for $p \in [2, \infty)$.

3.5 Define $g_n = (v * f)_n = \sum_{j=0}^n v_j d_j$. $S_n g = \left(\sum_{j=0} \tilde{d}_k\right)^{\frac{1}{2}}$. By previous exercise, triangle inequality for ℓ^2 norm and boundedness of v_n

$$\|g_n\|_p \leq C\|S_n g\|_p = C\|(\sum_{j=0}^{n} \tilde{d}_k^2)^{\frac{1}{2}}\|_p \leq C\|(\sum_{j=0}^{n} d_j^2)^{\frac{1}{2}}\|_p = C\|S_n f\|_p \leq C\|f_n\|_p$$

4. Up-crossing and convergence of martingales.

4.1 Let $\sum_{k=1}^{\infty} x_k$ be a series in Banach space. We want to show that if $\sum_{k=1}^{\infty} x_k$ converges unconditionally, then $\sum_{k=1}^{\infty} \delta_k x_k$ covnerges for every choice of $\delta_k \in \{0,1\}$

Let's reason by contradiction. Imagine there exists a family $\{\delta_k\}$ such that the series $\sum_{k=1}^{\infty} \delta_k x_k$ is divergent, which means that fails the Cauchy criterion. We fix an $\epsilon > 0$ so that the Cauchy criterion fails with $\epsilon > 0$, namely, there exists arbitrarily large $m < n$ such that $\| \sum_{k=m}^{n} \delta_k x_k \|_X \geq \epsilon$. We then can find a sequence of numbers $m_1 < n_1 < m_2 < n_2 < ...$ such that $\| \sum_{k=m_j}^{n_j} \delta_k x_k \|_X \geq \epsilon$.

Define $s_1(j) = \{k \in [m_j, n_j] : \delta_k = 1\}$, $s_0(j) = \{k \in [m_j, n_j] : \delta_k = 0\}$ and $t_j = \# s_1(j)$

Define a permutation as follows

$$\sigma(k) = \begin{cases} k & \text{if } \ k < m_1 \ or \ \exists j \in \mathbb{N} \ s.t. \ n_{j-1} < k < m_j \\ \min\{m \in s_1(j) : \sigma(k-1) < m\} & \text{if } k \in [m_j, m_j + t_j) \\ \min\{m \in s_0(j) : \sigma(k-1) < m\} & \text{if } k \in (m_j + t_j, n_j] \end{cases}$$

Clearly $\sigma(k)$ is a permutation, and the series $\{x_{\sigma(k)}\}$ fails the Cauchy criterium since $\| \sum_{k=m_j}^{m_j+t_j} x_{\sigma(k)} \|_x \geq \epsilon$ for $j \in \mathbb{N}$ which is a contradiction with the assumption that $\sum_{k=1}^{\infty} x_k$ converges unconditionally.

4.2 Let $\sup_{n \in \mathbb{Z}} \|f_n\|_2^2 = M \leq \infty$

$$\sum_{k=m}^{n} \|d_k\|_2^2 = \| \sum_{k=m}^{n} d_k \|_2^2 = \|f_n - f_m\|_2^2 \leq 2M$$

That implies that $\sum_{k=-\infty}^{\infty} \|d_k\|_2^2$ is a convergent series, so it satisfies the Cauchy criterium, i.e $\forall \epsilon > 0 \ \exists N_\epsilon, M_\epsilon$ such that $\forall m_1, n_1 > N_\epsilon$ and $\forall m_2, n_2 < M_\epsilon$

$$\sum_{k=m_1}^{n_1} \|d_k\|_2^2 = \|f_{n_1} - f_{m_1}\|_2^2 < \epsilon$$
$$\sum_{k=m_2}^{n_2} \|d_k\|_2^2 = \|f_{n_2} - f_{m_2}\|_2^2 < \epsilon$$

$\Rightarrow f_n$ is a Cauchy sequence in L^2 which is a complete space, and that gives us the convergence in L^2 and existence of the limit in L^2 as n approaches $\pm\infty$.