MARTINGALES AND HARMONIC ANALYSIS

Solutions to exercises

2. Discrete-time martingales and Doob's inequality.

2.7 *Real valued case:* For all $p \in [1,\infty]$,

$$\|f\|_p = \sup\left\{\int fg\,d\mu \colon \|g\|_{p'} \le 1\right\}. \tag{1}$$

Proof. First step is to show that if $f \in L^p$, then (1) holds. We use Hölder's inequality to see that for all $g \in L^{p'}$,

$$\int fg\,d\mu \le \int |f||g|\,d\mu \le \|f\|_p\,\|g\|_{p'},$$

and thus,

$$\sup\left\{\int fg\,d\mu \colon \|g\|_{p'} \le 1\right\} \le \|f\|_p.$$

Note that this, in particular, shows that if $f \in L^p$ then the right hand side of (1) is finite and consequently, if the right hand side of (1) is infinite, then $f \notin L^p$.

For the reverse inequality, consider the three cases:

- Suppose $1 < p < \infty$. We may assume that $\|f\|_p > 0$ (since otherwise $f = 0$ a.e. and the claimed equality holds in the form $0 = 0$). Let $g_0 := \operatorname{sgn}(f)|f|^{p-1}/\|f\|_p^{p-1}$ where $\operatorname{sgn}(f)(x) := 1$ if $f(x) \ge 0$ and $\operatorname{sgn}(f) := -1$ if $f(x) < 0$, is the sign of $f(x)$. Then $\|g_0\|_{p'} = 1$, and we get
$$\sup\left\{\int fg\,d\mu \colon \|g\|_{p'} \le 1\right\} \ge \int fg_0\,d\mu = \|f\|_p,$$
since $f \cdot \operatorname{sgn}(f) = |f|$.
- Then consider $p = 1$ and $p' = \infty$. Let $g_0 := \operatorname{sgn}(f)$. Then $\|g\|_\infty = 1$ and
$$\sup\left\{\int fg\,d\mu \colon \|g\|_\infty \le 1\right\} \ge \int fg\,d\mu = \int |f|\,d\mu = \|f\|_1.$$
- Finally, consider $p = \infty$ and $p' = 1$. For every $\varepsilon > 0$ there exists a set $E \subset \Omega$ with $\mu(E) > 0$ and $|f| > \|f\|_\infty - \varepsilon$ on E. Note that we might have $\mu(E) = \infty$. However, by σ-finiteness, $E = \cup_{i=0}^\infty (E \cap E_i)$ where $\mu(E_i) < \infty$ for all i and the sets E_i are disjoint. Thus, $0 < \mu(E) = \sum_{i=0}^\infty \mu(E \cap E_i)$, and it follows that $0 < \mu(E \cap E_k) < \infty$ for some k. Denote $E \cap E_k =: E_\varepsilon$ and let $g_\varepsilon := \operatorname{sgn}(f)1_{E_\varepsilon}/\mu(E_\varepsilon)$. Then $\|g_\varepsilon\| = 1$ and
$$\sup\left\{\int fg\,d\mu \colon \|g\|_1 \le 1\right\} \ge \int fg_\varepsilon\,d\mu = \frac{1}{\mu(E_\varepsilon)}\int_{E_\varepsilon} |f|\,d\mu > \|f\|_\infty - \varepsilon \quad \text{for all } \varepsilon > 0,$$
so that
$$\sup\left\{\int fg\,d\mu \colon \|g\|_1 \le 1\right\} \ge \|f\|_\infty.$$

We have shown that (1) is true for every $1 \le p \le \infty$ and $f \in L^p$.

The second and final step is to show that if the right hand side of (1) is finite, then $f \in L^p$ (in which case the two quantities are equal). To this end, suppose that f is any measurable function. For each $n \in \mathbb{N}$, by σ-finiteness, we may pick measurable sets $E_1 \subset E_2 \subset \ldots \subset E_n \to \Omega$ with $\mu(E_n) < \infty$. Define $F_n := E_n \cap \{|f| \le n\} \to \Omega$ and

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$f_n := 1_{F_n} f$. Then $f_n \in L^p$ for each n (and all $1 \le p \le \infty$), and thus, for each f_n, there exists g_n such that

$$\|f_n\|_p = \int f_n g_n \, d\mu \quad \text{and} \quad \|g_n\|_{p'} = 1.$$

This implies that

$$\|f\|_p = \sup_n \|f_n\|_p = \sup \left\{ \int f(1_{F_n} g_n) \, d\mu \colon n \in \mathbb{N} \right\} \le \sup \left\{ \int fg \, d\mu \colon \|g\|_{p'} \le 1 \right\}.$$

Vector valued case: Use same ideas; for example, to show the estimate $\ge$, use Hölder's inequality, first with respect to sequences, then with respect to integrals to see that

$$\int \sum_k f_k(\omega) g_k(\omega) \, d\mu(\omega) \le \int \|(f_k(\omega))_k\|_{\ell^p} \|(g_k(\omega))_k\|_{\ell^{q'}} \, d\mu \le \|(f_k)\|_{L^p(\ell^q)} \|(g_k)\|_{L^{p'}(\ell^{q'})}.$$

For example, the vectors (g_k) that give us the equality

$$\|(f_k)\|_{L^p(\ell^q)} = \int \sum_k f_k g_k \, d\mu$$

are

$$g_k(\omega) := \operatorname{sgn}(f_k(\omega)) \cdot |f_k(\omega)|^{q-1} \cdot \|(f_k(\omega))\|_{\ell^q}^{p-q} \cdot \|(f_k)\|_{L^p(\ell^q)}^{1-p}$$

in the case $1 < p, g < \infty$ and

$$g_k(\omega) = \operatorname{sgn}(f_k(\omega)) \cdot \|(f_k(\omega))\|_{\ell^\infty}^{p-1} \cdot \|(f_k)\|_{L^p(\ell^\infty)}^{1-p}$$

for $k = k_0$ with $|f_{k_0}(\omega)| > (1-\varepsilon)\|(f_k(\omega))\|_{\ell^\infty}$ and $g_k(\omega) = 0$ for other k in the case $1 < p < \infty, q = \infty$.

□

Facts: In the solutions, we will repeatedly use the duality provided by Exercise 1 together with the following facts about the conditional expectation:

Fact 1: The operator $f \mapsto \mathbb{E}[f|\mathscr{G}]$ is a contraction: $\|\mathbb{E}[f|\mathscr{G}]\|_p \le \|f\|_p$ for all $1 \le p \le \infty$. *Proof:* Corollary 1.16

Fact 2: $\mathbb{E}[g \cdot f|\mathscr{G}] = g \cdot \mathbb{E}[f|\mathscr{G}]$ for $g \in L^0(\mathscr{G})$. *Proof:* Theorem 1.20

Fact 3: The operator $f \mapsto \mathbb{E}[f|\mathscr{G}]$ is self-adjoint:

$$\int_\Omega \mathbb{E}[f|\mathscr{G}] \cdot g \, d\mu = \int_\Omega f \cdot \mathbb{E}[g|\mathscr{G}] \, d\mu.$$

Proof: Use the definition of conditional expectation and Fact 2.

2.8 Let $(\mathscr{F}_k)_{k \in \mathbb{Z}}$ be a filtration and $f \in L^p(\mathscr{F})$, and abbreviate $E_k f := \mathbb{E}[f|\mathscr{F}_k]$.

$(*) \Rightarrow$ *Doob:* Let $f = (f_i)_{i \in \mathbb{Z}}$ be a nonnegative submartigale with $\sup_{i \in \mathbb{Z}} \|f_i\|_p < \infty$. We wish to bound the quantity

$$\|f^*\|_p = \left\| \sup_i |f_i| \right\|_p = \|(f_i)\|_{L^p(\ell^\infty)} = \sup \left\{ \int \sum_i f_i g_i \, d\mu \colon \|(g_i)\|_{L^{p'}(\ell^1)} \le 1 \right\}.$$

For any such (g_i), we have that

$$\int \sum_i f_i g_i \, d\mu = \lim_{n \to \infty} \int \sum_{i \le n} f_i |g_i| \, d\mu$$

Here, for every n, since (f_i) is a submartingale and by Fact 3 and Hölder's inequality,

$$\begin{aligned}\int \sum_{i\le n} f_i|g_i|\,d\mu &\le \int \sum_{i\le n} E_i f_n |g_i|\,d\mu = \int f_n \sum_{i\le n} E_i|g_i|\,d\mu \\ &\le \|f_n\|_p \cdot \Big\| \sum_i E_i|g_i| \Big\|_{p'} \le p' \cdot \sup_k \|f_k\|_p \cdot \Big\| \sum_i |g_i| \Big\|_{p'} \\ &= p' \cdot \sup_k \|f_k\|_p \cdot \|(g_i)\|_{L^{p'}(\ell^1)} = p' \cdot \sup_k \|f_k\|_p\end{aligned}$$

where we used $(*)$ (in $L^{p'}$) in the last estimate, and the upper bound does not depend on n. Doob's Inequality follows by taking supremum over all relevant (g_i).

To prove *Doob* $\Rightarrow (*)$, we consider the following *version of the Doob's Inequality*, which is a version of the slightly more general case, Theorem 2.10 in the lecture notes:

Let $f \in L^p$ (so, f is a function, not a sequence or a martingale). Let $(\mathscr{F}_k)_k$ be a filtration, and denote $E_k f := \mathbb{E}[f|\mathscr{F}_k]$. Set

$$Mf := \sup_{k\in\mathbb{Z}} |E_k f| = \|(E_k f)\|_{\ell^\infty}. \tag{M}$$

Then

$$\|Mf\|_p \le p' \cdot \|f\|_p \quad \text{for all } 1 < p \le \infty.$$

To convince oneself that this is in the regime of Theorem 2.10, note that $(E_k f)$ is a martingale for $f \in L^p$, and thus, $(E_k|f|)$ is a nonnegative submartingale, and make use of Fact 1.

Doob $\Rightarrow (*)$*:* By duality in L^p,

$$\Big\| \sum_k E_k f_k \Big\|_p = \sup \left\{ \int \left(\sum_k E_k f_k \right) g\,d\mu \colon \|g\|_{p'} \le 1 \right\}.$$

For any such g, we have that

$$\int \sum_k E_k f_k g\,d\mu = \int \sum_k f_k E_k g\,d\mu \le \int Mg \sum_k f_k\,d\mu \le \Big\| \sum_k f_k \Big\|_p \|Mg\|_{p'} \le p \cdot \Big\| \sum_k f_k \Big\|_p$$

where we used Fact 3, Hölder's inequality and the version of Doob's inequality. Taking supremum over all relevant g, $(*)$ follows.

2.9 We wish to bound the quantity

$$\Big\| \sum_k E_k f_k \Big\|_2^2 = \int \left(\sum_k E_k f_k \right)^2 d\mu.$$

We estimate

$$\begin{aligned}\int \left(\sum_k E_k f_k \right)^2 d\mu &= \int \left(\sum_k (E_k f_k)^2 + 2 \sum_\ell \sum_{k<\ell} E_k f_k \cdot E_\ell f_\ell \right) d\mu \le 2 \int \sum_\ell \sum_{k\le\ell} E_k f_k \cdot E_\ell f_\ell\,d\mu \\ &= 2 \sum_\ell \sum_{k\le\ell} \int E_\ell(E_k f_k \cdot f_\ell)\,d\mu = 2 \sum_\ell \sum_{k\le\ell} \int E_k f_k \cdot f_\ell\,d\mu \\ &\le 2 \int \sum_k E_k f_k \cdot \sum_\ell f_\ell\,d\mu \overset{3.}{\le} 2 \Big\| \sum_k E_k f_k \Big\|_2 \cdot \Big\| \sum_\ell f_\ell \Big\|_2;\end{aligned}$$

where we used Fact 2 together with the summation condition $k \le \ell$, the definition of conditional expectation and Cauchy-Schwarz. We obtained

$$\Big\| \sum_k E_k f_k \Big\|_2^2 \le 2 \Big\| \sum_k E_k f_k \Big\|_2 \cdot \Big\| \sum_k f_k \Big\|_2$$

To complete the proof, we wish to divide by $\left\|\sum_k E_k f_k\right\|_2$ and to do this, we must have a finite quantity. Note that for a truncated summation we have, by Minkowski's inequality and fact 1, that

$$\left\|\sum_{|k|\leq N} E_k f_k\right\|_2 \leq \sum_{|k|\leq N} \left\|E_k f_k\right\|_2 \leq \sum_{|k|\leq N} \|f_k\|_2 < \infty.$$

Thus, $(*)$ is true for any truncated sum with an upper bound does not depend on N, the length of the summation. The monotone convergence theorem then completes the proof.

3. Martingale transforms and Burkholder's inequality.

3.1 Recall from Section 1.9 in the lecture notes that for $f \in L^2(\mathscr{F}, \mu)$, $\mathbb{E}[f|\mathscr{F}_k] \in L^2(\mathscr{F}_k, \mu)$ is the orthogonal projection of f onto the closed subspace $L^2(\mathscr{F}_k, \mu) \subseteq L^2(\mathscr{F}, \mu)$, and hence,

$$(f - \mathbb{E}[f|\mathscr{F}_k]) \perp L^2(\mathscr{F}_k, \mu).$$

Let $(f_k)_{k=0}^n$ be an L^2- martingale adapted to a filtration $(\mathscr{F}_k)_{k=0}^n$ and let $(v_k)_{k=0}^n$ be a bounded predictable sequence with $\|v_k\|_\infty \leq 1$. By martingale property,

$$d_k = (f_k - f_{k-1}) = (f_k - \mathbb{E}[f_k|\mathscr{F}_{k-1}]) \perp L^2(\mathscr{F}_{k-1}, \mu) \supseteq \{d_0, d_1, \ldots, d_{k-1}\} \quad \forall\, 1 \leq k \leq n.$$

Consequently, $\{d_k\}_{k=0}^n \subseteq L^2(\mathscr{F}, \mu)$ is an orthogonal set (i.e. $\int d_i \cdot d_j \, d\mu = 0$ for all $i \neq j$). It is easy to check that also $\{v_k d_k\}_{k=0}^n$ is an orthogonal set. By applying Pythagorean Theorem twice (or Minkowski's inequality and Pythagorean Theorem), and by the facts that $\|v_k\|_\infty \leq 1$ and $f_n = \sum_{k=0}^n d_k$, we obtain

$$\|(v * f)_n\|_2^2 = \left\|\sum_{k=0}^n v_k d_k\right\|_2^2 = \sum_{k=0}^n \|v_k d_k\|_2^2 \leq \sum_{k=0}^n \|d_k\|_2^2 = \left\|\sum_{k=0}^n d_k\right\|_2^2 = \|f_n\|_2^2$$

and hence $\|(v * f)_n\|_2 \leq \|f_n\|_2$.