MARTINGALES AND HARMONIC ANALYSIS

2. Solutions to exercises

2.1. Discrete-time martingales and Doob's inequality.

1. We may assume that $I \neq \varnothing$. For all $j \in \mathbb{Z}$, let $N(j) := \{i \in I : i \geq j\}$. This is empty for some j if and only if I is bounded from above, i.e., there exists $\max I$. Set $n(j) := \min N(j)$ if $N(j) \neq \varnothing$, and $n(j) := \max I$ otherwise. With the help of the tower rule one easily checks that $\mathscr{F}_j := \mathscr{F}_{n(j)}$ is a filtration and $f_j := f_{n(j)}$ a martingale adapted to it. Clearly $n(i) = i$ if $i \in I$, so this is an extension of the original one.
2. For the filtration of the hint, there holds
$$f_{-n}(x) := \mathbb{E}[f|\mathscr{F}_{-n}](x) = \frac{1}{n\delta}\int_0^{n\delta} f(y)\,\mathrm{d}y, \qquad x \in (0, n\delta],$$
so the corresponding Doob's maximal function satisfies, for $x \in ((n-1)\delta, n\delta]$,
$$Mf(x) \geq \frac{1}{n\delta}\int_0^{n\delta} f(y)\,\mathrm{d}y \geq \frac{1}{x+\delta}\int_0^{x} f(y)\,\mathrm{d}y =: F_\delta(x).$$
One can easily check that $\|f_i\|_p \leq \|f\|_p$ for all $i \in \mathbb{Z}_-$, then by Doob's inequality $\|F_\delta\|_p \leq p'\|f\|_p$, and the claim follows from monotone convergence as $\delta \searrow 0$.
3. Let $f(x) := 1_{(0,1]}(x) \cdot x^\alpha$. This is in $L^p(\mathbb{R}_+)$ if and only if $\alpha > -1/p$. Now
$$F(x) := \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y = \frac{x^\alpha}{1+\alpha} = (1+\alpha)^{-1}f(x), \qquad x \in (0,1].$$
where $1+\alpha > 1/p' > 0$. Hence, if Hardy's inequality holds with some constant C, then $C \geq \|F\|_p/\|f\|_p \geq (1+\alpha)^{-1}$ $\forall \alpha > -1/p$. In the limit $\alpha \searrow -1/p$, there follows that $C \geq (1-1/p)^{-1} = p'$.
4. We define $\mathscr{F}_k := \sigma(\mathscr{D}_k^\beta)$. One has to show that $\mathscr{F}_k \subseteq \mathscr{F}_{k+1}$. It suffices to prove that every $J \in \mathscr{D}_k^\beta$ is a (necessarily countable) union of some sets in $\mathscr{D}_{k+1}^\beta$. By definition, $J = I + \sum_{j>k} 2^{-j}\beta_j$ for some $I = 2^{-k}[\ell, \ell+1) \in \mathscr{D}_k$. Clearly $I = I_0 \cup I_1$, where
$$I_i := 2^{-k}[\ell + i/2, \ell + (i+1)/2) = 2^{-(k+1)}[2\ell + i, 2\ell + i + 1) \in \mathscr{D}_{k+1}.$$
Now $J = J_0 \cup J_1$, if we set
$$\begin{aligned} J_i := I_i + \sum_{j>k} 2^{-j}\beta_j &= (I_i + 2^{-(k+1)}\beta_{k+1}) + \sum_{j>k+1} 2^{-j}\beta_j \\ &\in \mathscr{D}_{k+1} + \sum_{j>k+1} 2^{-j}\beta_j = \mathscr{D}_{k+1}^\beta. \end{aligned}$$
5. Consider the endpoints of the intervals $I \in \mathscr{D}_k^0 \cup \mathscr{D}_k^\beta$. They have the form $2^{-k}\ell$ or $2^{-k}\ell + \sum_{j>k} 2^{-j}\beta_j = 2^{-k}\big(\ell + \sum_{j>k} 2^{k-j}\beta_j\big)$, where $\ell \in \mathbb{Z}$. Depending on the parity of k, there holds one of
$$\sum_{j>k} 2^{k-j}\beta_j = \begin{cases} \sum_{j=1,3,5,\ldots} 2^{-j} = 2^{-1}\sum_{j=0}^\infty 4^{-j} = 2/3, \\ \sum_{j=2,4,5,\ldots} 2^{-j} = 4^{-1}\sum_{j=0}^\infty 4^{-j} = 1/3. \end{cases}$$
Thus the endpoints have the form $2^{-k}\ell$ and either $2^{-k}(\ell + 1/3)$ or $2^{-k}(\ell + 2/3)$; in either case, the minimal distance of two endpoints is $2^{-k}/3$.

   Let then J be some finite subinterval of $\mathbb{R}$. Choose the unique $k \in \mathbb{Z}$ with $3|J| < 2^{-k} \leq 6|J|$. Since $|J| < 2^{-k}/3$, the interval J can contain at most one endpoint of one interval

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$I' \in \mathscr{D}_k^0 \cup \mathscr{D}_k^\beta$. If $I' \in \mathscr{D}_k^0$, then J does not contain any endpoints of intervals in $\mathscr{D}_k^\beta$. On the other hand, $\mathscr{D}_k^\beta$ covers all of $\mathbb{R}$, so in particular all of J. Since J is a connected interval and does not contain endpoints of $\mathscr{D}_k^\beta$, it must be completely contained in a single interval $I \in \mathscr{D}_k^\beta$. Symmetrically, if $I' \in \mathscr{D}_k^\beta$, then there exists $I \in \mathscr{D}_k^0$, which contains J.

In any case $J \subset I \in \mathscr{D}_k^0 \cup \mathscr{D}_k^\beta$ and $|I| = 2^{-k} \leq 6|J|$.

6. Let M^0 and M^β be Doob's maximal operators related to the filtrations $(\mathscr{F}_k^0)_{k\in\mathbb{Z}} := (\sigma(\mathscr{D}_k^0))_{k\in\mathbb{Z}}$ and $(\mathscr{F}_k^\beta)_{k\in\mathbb{Z}} := (\sigma(\mathscr{D}_k^\beta))_{k\in\mathbb{Z}}$ (i.e. $(f_k^0)_{k\in\mathbb{Z}} := (\mathbb{E}[|f||\mathscr{F}_k^0])_{k\in\mathbb{Z}}$ and $(f_k^\beta)_{k\in\mathbb{Z}} := (\mathbb{E}[|f||\mathscr{F}_k^\beta])_{k\in\mathbb{Z}}$). Let $x \in \mathbb{R}$ and $J \ni x$ be a finite subinterval of $\mathbb{R}$. By the previous exercise, there exists $I \in \mathscr{D}^0 \cup \mathscr{D}^\beta$, such that $I \supset J$ and $|I| \leq 6|J|$. Hence
$$\frac{1}{|J|}\int_J |f(y)|\,\mathrm{d}y \leq \frac{6}{|I|}\int_I |f(y)|\,\mathrm{d}y \leq \begin{cases} 6M^0|f|(x), & I \in \mathscr{D}^0, \\ 6M^\beta|f|(x), & I \in \mathscr{D}^\beta. \end{cases}$$
Taking the supremum over all $J \ni x$ on the left, we obtain
$$M_{HL}f(x) \leq 6M^0|f|(x) + 6M^\beta|f|(x)$$
and then by Doob's inequality
$$\|M_{HL}f\|_p \leq 6\|M^0|f|\,\|_p + 6\|M^\beta|f|\,\|_p \leq 12p'\|f\|_p.$$