MARTINGALES AND HARMONIC ANALYSIS

1. Solutions to exercises

1.1. Conditional expectation.

1. E.g. $\Omega = \mathbb{R}$, $\mathscr{F}$ = Borel σ-algebra, $\mathscr{G} = \{\varnothing, \mathbb{R}\}$ and μ = Lebesgue measure. Another possibility is e.g. $\mathscr{G} = \{\varnothing, (-\infty, 0), [0, \infty), \mathbb{R}\}$, which shows that $\mathscr{G}$ need not be the trivial σ-algebra.
2. Denote $h_n := 2g - |f - f_n|$. Then $0 \le h_n \in L^1_{\mathscr{G}^0}(\mathscr{F}, \mu)$ and
$$\liminf_{n\to\infty} h_n = \lim_{n\to\infty} h_n = 2g \in L^1_{\mathscr{G}^0}(\mathscr{F}, \mu).$$
By the conditional Fatou's lemma
$$0 \le \mathbb{E}[2g|\mathscr{G}] \le \liminf_{n\to\infty} \mathbb{E}[h_n|\mathscr{G}] = \mathbb{E}[2g|\mathscr{G}] - \limsup_{n\to\infty} \mathbb{E}[|f - f_n|\,|\mathscr{G}],$$
which implies $\limsup_{n\to\infty} \mathbb{E}[|f - f_n|\,|\mathscr{G}] \le 0$, and thus, $\limsup_{n\to\infty} \mathbb{E}[|f - f_n|\,|\mathscr{G}] = 0$. Consequently, also the limit exists and it is equal to 0. Finally note that $0 \le |\mathbb{E}[f_n|\mathscr{G}] - \mathbb{E}[f|\mathscr{G}]| \le \mathbb{E}[|f_n - f|\,|\mathscr{G}]$, and we may conclude that $\exists \lim_{n\to\infty} |\mathbb{E}[f_n|\mathscr{G}] - \mathbb{E}[f|\mathscr{G}]| = 0$.
3. Let $H \in \mathscr{H}^0 \subseteq \mathscr{G}^0$. Using the definition of the conditional expectation three times, we see that
$$\int_H \mathbb{E}\big(\mathbb{E}[f|\mathscr{G}]\big|\mathscr{H}\big)\,\mathrm{d}\mu = \int_H \mathbb{E}[f|\mathscr{G}]\,\mathrm{d}\mu = \int_H f\,\mathrm{d}\mu = \int_H \mathbb{E}[f|\mathscr{H}]\,\mathrm{d}\mu,$$
and Lemma 1.3 completes the argument. In fact, the first two equalities above show that $\mathbb{E}\big(\mathbb{E}[f|\mathscr{G}]\big|\mathscr{H}\big)$ is (one possible) conditional expectation of f. By recalling that it is unique, this shows the claim, and we do not need to use Lemma 1.3.
4. The familiar inequality from the hint follows by using the property $\phi(x/p + y/p') \le \phi(x)/p + \phi(y)/p'$ applied to the convex function $\phi(x) = e^x$ and the values $x := \log a^p$, $y =: \log b^{p'}$. Another possibility is to move all the terms on one side of the inequality and investigate the resulting expression, say, as a function of $a \in [0, \infty)$ for a fixed b. Checking the non-negativity is a high school level exercise in differentiation — the extremal values are reached at the endpoints of the interval or at the zeros of the derivative.

   From the hint it follows directly that
$$|\mathbb{E}[f \cdot g|\mathscr{G}]| \le \mathbb{E}[|f| \cdot |g|\,|\mathscr{G}] \le \frac{1}{p}\mathbb{E}[|f|^p|\mathscr{G}] + \frac{1}{p'}\mathbb{E}[|g|^{p'}|\mathscr{G}].$$
If one replaces f by the function $\lambda \cdot f$ and g by g/λ, where $\lambda > 0$ is a constant, the left side of the previous inequality stays invariant, but the right side becomes
$$\frac{\lambda^p}{p}\mathbb{E}[|f|^p|\mathscr{G}] + \frac{\lambda^{-p'}}{p'}\mathbb{E}[|g|^{p'}|\mathscr{G}] =: \frac{\lambda^p}{p}F + \frac{\lambda^{-p'}}{p'}G.$$
Thus, the estimate
$$|\mathbb{E}[f \cdot g|\mathscr{G}]| \le \frac{\lambda^p}{p}F + \frac{\lambda^{-p'}}{p'}G$$
holds for all $\lambda > 0$. We make two observations: First, if $F = 0$, then by taking the limit $\lambda \to \infty$ we see that $\mathbb{E}[f \cdot g|\mathscr{G}] = 0$, and the claim follows. Second, if $G = 0$, then by taking the limit $\lambda \to 0$ we see that $\mathbb{E}[f \cdot g|\mathscr{G}] = 0$, and again the claim follows. Then assume that $F, G > 0$. We minimize the expression on the right side with respect to λ at each $\omega \in \Omega$ by setting $\lambda = (G/F)^{1/(pp')} \in (0, 1)$; this particular λ satisfies $\lambda^p F = \lambda^{-p'} G$. The claimed upper bound for $\mathbb{E}[f \cdot g|\mathscr{G}]$ follows.

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5. We first observe that $\tilde{\mathscr{B}} = \{B \subseteq \mathscr{B} \colon x \in B \Rightarrow -x \in B\}$. To see that $\tilde{\mathscr{B}}$ is a σ-algebra, first let $B \in \tilde{\mathscr{B}}$. If $x \in B^c$, then $-x \in B^c$ since otherwise $-x \in B \Rightarrow x = -(-x) \in B$, a contradiction. This shows that $B^c \in \tilde{\mathscr{B}}$. Second, let $E_i \in \tilde{\mathscr{B}}$ for $i = 1, 2, \ldots$. If $x \in \bigcup_{i=1}^{\infty} E_i$, then $x \in E_i$ and so $-x \in E_i$ for some i. Thus, $-x \in \bigcup_{i=1}^{\infty} E_i$ and consequently $\bigcup_{i=1}^{\infty} E_i \in \tilde{\mathscr{B}}$. Finally, make the (trivial) observations that $\emptyset, \mathbb{R} \in \tilde{\mathscr{B}}$.

Let $f \in L^1_{\mathscr{B}^0}(\mathbb{R}, \mathscr{B}, dx)$, and denote $\tilde{f} := (f(x) + f(-x))/2$. To see that $\tilde{f} = \mathbb{E}[f|\tilde{\mathscr{B}}](x)$, we need to show that (1) $\tilde{f}$ is $\tilde{\mathscr{B}}$-measurable, and (2) $\int_B \tilde{f}\, d\mu = \int_B f\, d\mu$ for all $B \in \tilde{\mathscr{B}}^0$. For (1), suppose $B \in \mathscr{B}(\mathbb{R})$ is a Borel set and $y \in \tilde{f}^{-1}(B) = \{x \in \mathbb{R} \colon \tilde{f}(x) \in B\}$. Since $\tilde{f}(-y) = \tilde{f}(y) \in B$, also $-y \in \tilde{f}^{-1}(B)$ and consequently $\tilde{f}^{-1}(B) \in \tilde{\mathscr{B}}$.

For (2), suppose $B \in \tilde{\mathscr{B}}^0$. We calculate

$$\begin{aligned}\int_B \tilde{f}(x)\, dx &= \frac{1}{2}\left(\int_B f(x)\, dx + \int_B f(-x)\, dx\right) \\ &= \frac{1}{2}\left(\int_B f(x)\, dx + \int_{-B} f(x)\, dx\right) = \frac{1}{2} \cdot 2 \cdot \int_B f(x)\, dx = \int_B f(x)\, dx\end{aligned}$$

since $-B = B$.