Lecture 11: periodic orbits as limit sets

Introduction and notation

The expounded material can be found in

- Chapter 2 of [1]
- Chapter 3 of [2]

As usual we suppose that

$$\dot{\phi}_t = \boldsymbol{f} \circ \phi_t \tag{0.1}$$

is driven by a vector field **sufficiently smooth** to guarantee the existence of a flow $\boldsymbol{\Phi} : \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\phi_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

1 Definition and basic properties

Definition 1.1. *A set $\mathcal{S} \in \mathcal{D}$ is called*

- *positively invariant if*

$$\boldsymbol{\Phi}_t (\mathcal{S}) \subset \mathcal{S} \qquad \forall\, t \geq 0 \tag{1.1}$$

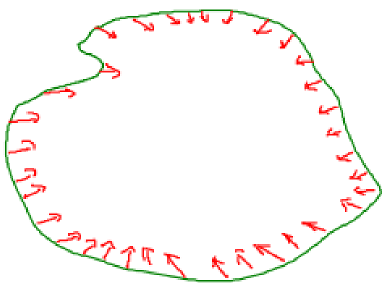
- *negatively invariant if*

$$\boldsymbol{\Phi}_t (\mathcal{S}) \subset \mathcal{S} \qquad \forall\, t \leq 0 \tag{1.2}$$

- *invariant if*

$$\boldsymbol{\Phi}_t (\mathcal{S}) = \mathcal{S} \qquad \forall\, t \leq 0 \tag{1.3}$$

A positively invariant set can be constructed in two dimensions



(1.4)

when it is possible to identify a closed curve along which the vector field $\boldsymbol{f}$ is always pointing towards the interior of the area encompassed by the curve.

Definition 1.2. *We call the*

- *ω-limit set of any $\boldsymbol{x} \in \mathcal{D}$ the set*

$$\omega(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathcal{D} \,|\, \lim_{n\uparrow\infty} \boldsymbol{\Phi}_{t_n}(\boldsymbol{x}) = \boldsymbol{y} \right\} \tag{1.5}$$

for some sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_n \stackrel{n\uparrow\infty}{\to} \infty$.

- *α-limit set of any $\boldsymbol{x} \in \mathcal{D}$ the set*

$$\omega(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathcal{D} \,|\, \lim_{n\uparrow\infty} \boldsymbol{\Phi}_{t_n}(\boldsymbol{x}) = \boldsymbol{y} \right\} \tag{1.6}$$

for some sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_n \stackrel{n\uparrow\infty}{\to} -\infty$.

Obvious examples of ω (α) sets are asymptotically stable (unstable) fixed point.

Proposition 1.1. *Let $\mathcal{S}$ positively invariant and compact. Then for any $\boldsymbol{x} \in \mathcal{S}$, $\omega(\boldsymbol{x})$ enjoys the following properties*

1. *it is not empty: $\omega(\boldsymbol{x}) \neq \emptyset$;*
2. *it is closed;*
3. *it is invariant under the flow: $\boldsymbol{\Phi}_t \circ \omega(\boldsymbol{x}) = \omega \circ \boldsymbol{\Phi}_t(\boldsymbol{x})$ for any t;*
4. *it is connected;*

Proof.

1. Let some sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_n \stackrel{n\uparrow\infty}{\to} \infty$ and define

$$\boldsymbol{x}_n := \boldsymbol{\phi}_{t_n}(\boldsymbol{x}) \tag{1.7}$$

Since $\mathcal{S}$ is compact, it is always possible to extract a convergent sub-sequence $\{t_{n_k}\}_{k=0}^{\infty}$. By definition

$$\lim_{k\uparrow\infty} \boldsymbol{\Phi}_{t_{n_k}}(\boldsymbol{x}) \in \omega(\boldsymbol{x}) \tag{1.8}$$

2. Let suppose $\boldsymbol{x}_1 \notin \omega(\boldsymbol{x})$. Then there must be a neighborhood $\mathcal{U}$ of $\boldsymbol{x}_1$ and a $t_1 > 0$ such that $\boldsymbol{\Phi}_t(\boldsymbol{x}) \notin \mathcal{U}$ for any $t \geq t_1$. But this is equivalent to say that $\bar{\omega}(\boldsymbol{x})$ is open. Hence $\omega(\boldsymbol{x})$ must be closed.

3. Let $\boldsymbol{y} \in \omega(\boldsymbol{x})$ then by definition there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_n \stackrel{n\uparrow\infty}{\to} \infty$ (Note: we can always choose such sequence to be monotone by extracting an increasing sub-sequence from the convergent sequence obtained using compactness) for which

$$\lim_{n\uparrow\infty} \boldsymbol{\Phi}_{t_n}(\boldsymbol{x}) = \boldsymbol{y} \tag{1.9}$$

For any fixed n we can find a $t \in \mathbb{R}$ such that

$$t + t_n \geq 0 \tag{1.10}$$

From the properties of the flow we must have

$$\mathbf{\Phi}_{t+t_n}(\boldsymbol{x}) = \begin{cases} \mathbf{\Phi}_t \circ \mathbf{\Phi}_{t_n}(\boldsymbol{x}) \\ \mathbf{\Phi}_{t_n} \circ \mathbf{\Phi}_t(\boldsymbol{x}) \end{cases} \tag{1.11}$$

Passing to the limit $n \uparrow \infty$ we obtain the identity

$$\mathbf{\Phi}_t \circ \omega(\boldsymbol{x}) = \omega \circ \mathbf{\Phi}_t(\boldsymbol{x}) \tag{1.12}$$

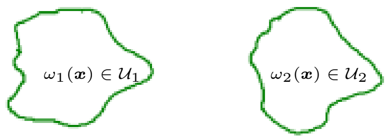
which yields the claim.

4. The proof proceeds by "*reductio ad absurdum*". Let us suppose

$$\omega(\boldsymbol{x}) = \omega_1(\boldsymbol{x}) \cup \omega_1(\boldsymbol{x}) \tag{1.13}$$

with $\omega_i(\boldsymbol{x}) \subset \mathcal{U}_i$, $i = 1.2$ and

$$\bar{\mathcal{U}}_1 \cap \bar{\mathcal{U}}_2 = \emptyset \tag{1.14}$$



By continuity of $\mathbf{\Phi}_t(\cdot)$ given a $t_o > 0$ there exists a $t > t_o$ such that

$$\mathbf{\Phi}_t(\boldsymbol{x}) \in \mathcal{S}/(\mathcal{U}_1 \cup \mathcal{U}_1) \tag{1.15}$$

hence we can construct a monotonically increasing sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$\mathbf{\Phi}_{t_n}(\boldsymbol{x}) \in \mathcal{S}/(\mathcal{U}_1 \cup \mathcal{U}_1) \tag{1.16}$$

Exploiting now the compactness of $\mathcal{S}$ we can always extract a sub-sequence convergent to some $\boldsymbol{y} \in \mathcal{S}/(\mathcal{U}_1 \cup \mathcal{U}_1)$. But in such a case $\boldsymbol{y} \in \omega(\boldsymbol{x})$ by definition and we therefore reached a contradiction.

□

2 Periodic Orbits

Definition 2.1. *A periodic orbit is an orbit forming a closed curve O in $\mathcal{D}$. Equivalently, if $\boldsymbol{x} \in \mathcal{D}$ is not an equilibrium point, and*

$$\mathbf{\Phi}_T(\boldsymbol{x}_o) = \boldsymbol{x}_o \tag{2.1}$$

for some $T > 0$, then the orbit of $\boldsymbol{x}_o$ is a periodic orbit with period T if furthermore

$$\mathbf{\Phi}_t(\boldsymbol{x}_o) \neq \boldsymbol{x}_o \tag{2.2}$$

for all $0 < t < T$.

Let $\phi_t = \mathbf{\Phi}_t(\boldsymbol{x})$ be a trajectory on a periodic orbit O. By Floquet theorem, the linearized flow

$$\mathsf{F}_t := \partial_{\boldsymbol{x}} \otimes \mathbf{\Phi}_t(\boldsymbol{x}) \tag{2.3}$$

factorizes as

$$\mathsf{F}_t = \mathsf{P}_t \, e^{\mathsf{B}\,t} \tag{2.4}$$

with $\mathsf{P}_{t+T} = \mathsf{P}_t$ for all t. Since

$$\dot{\boldsymbol{\phi}}_t = \mathsf{F}_t \cdot \dot{\boldsymbol{\phi}}_0 \tag{2.5}$$

it follows that the monodromy matrix

$$\mathsf{M} := e^{\mathsf{B}T} \tag{2.6}$$

always admit one unit eigenvalue in consequence of the chain of equalities

$$\mathsf{M} \cdot \dot{\boldsymbol{\phi}}_0 = \dot{\boldsymbol{\phi}}_T = \dot{\boldsymbol{\phi}}_0 \tag{2.7}$$

Proposition 2.1. *The eigenvalues of the monodromy matrix depend only upon the periodic obit and not upon individual trajectories on it*

Proof. Let us compare two trajectories on the periodic orbit:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_{t_1}(\boldsymbol{x}) \tag{2.8}$$

and

$$\tilde{\boldsymbol{\phi}}_t = \boldsymbol{\Phi}_t(\tilde{\boldsymbol{x}}) \tag{2.9}$$

By the our hypothesis there must exist a t' such that

$$\tilde{\boldsymbol{\phi}}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_{\tilde{t}}(\boldsymbol{x}) = \boldsymbol{\Phi}_{t+\tilde{t}}(\boldsymbol{x}) \tag{2.10}$$

If we denote by F and $\tilde{\mathsf{F}}$ the linearization of the flows we obtain

$$\mathsf{F}_{t+\tilde{t}} = \tilde{\mathsf{F}}_t \mathsf{F}_{\tilde{t}} \qquad \Rightarrow \qquad \tilde{\mathsf{F}}_t = \mathsf{F}_{t+\tilde{t}} \mathsf{F}_{\tilde{t}}^{-1} \tag{2.11}$$

which implies that the matrices satisfy the similarity transformation

$$\tilde{\mathsf{M}} = \mathsf{P}_{T+\tilde{t}} \, e^{\mathsf{B}(T+\tilde{t})} \, e^{-\mathsf{B}\tilde{t}} \, \mathsf{P}_{\tilde{t}}^{-1} = \mathsf{P}_{\tilde{t}} \mathsf{M} \, \mathsf{P}_{\tilde{t}}^{-1} \tag{2.12}$$

The characteristic polynomial is therefore independent of the trajectory chosen

$$\det(\tilde{\mathsf{M}} - \lambda \, 1) = \det(\mathsf{P}_{\tilde{t}} \mathsf{M} \, \mathsf{P}_{\tilde{t}}^{-1} - \lambda \, 1) = \det(\mathsf{M} - \lambda \, 1) \tag{2.13}$$

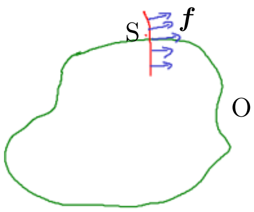
□

2.1 Poincaré map

Let $\boldsymbol{x}_\star$ be a point on a periodic orbit O.

Definition 2.2. *A Poincaré section* S *is a differentiable sub-manifold of co-dimension one (i.e. if we are in* d *dimensions it is a* $\mathrm{d}-1$ *dimensional set)* <u>*containing*</u> $\boldsymbol{x}_\star$ *and* <u>*transverse*</u> *to the flow of (0.1). This means that if we denote by* $\boldsymbol{n}(\boldsymbol{x})$ *a vector field perpendicular to* S *for any* $\boldsymbol{x} \in \mathrm{S}$ *then*

$$\boldsymbol{n} \cdot \boldsymbol{f} \neq 0 \qquad \forall \boldsymbol{x} \in \mathrm{S} \tag{2.14}$$



Given a Poincaré section we can define the first return time of the flow $\boldsymbol{\Phi}$ of $\boldsymbol{f}$ to S as the map

$$\tau \colon \mathrm{S} \mapsto (0, +\infty] \tag{2.15}$$

defined by

$$\tau(\boldsymbol{x}) = \inf \{ t > 0 \,|\, \boldsymbol{x} \in \mathrm{S} \;\; \& \;\; \boldsymbol{\Phi}_t(\boldsymbol{x}) \in \mathrm{S} \} \in \tag{2.16}$$

The return time is equal to infinity if the flow of $\boldsymbol{x}$ does not return to S at any further finite time. The existence of the return time is guaranteed by the following proposition:

Proposition 2.2. *Let* H *be an hyper-plane orthogonal to* O *at* $\boldsymbol{x}_\star$*:*

$$\mathrm{H} := \Big\{ \boldsymbol{x} \in \mathbb{R}^d \,|\, (\boldsymbol{x} - \boldsymbol{x}_\star) \cdot \boldsymbol{f}(\boldsymbol{x}_\star) = 0 \Big\} \tag{2.17}$$

Then there is a $\delta > 0$ *and a unique function* $\tau(\boldsymbol{x})$*, defined and continuously differentiable for any* $\boldsymbol{x}$ *in a ball* $\mathrm{B}_\delta(\boldsymbol{x}_\star)$ *of radius* δ *centered at* $\boldsymbol{x}_\star$*, such that*

- $\tau(\boldsymbol{x}_\star) = T$*;*
- $\boldsymbol{\Phi}_{\tau(\boldsymbol{x})}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathrm{B}_\delta(\boldsymbol{x}_\star)$

Proof. The proof relies on the implicit function theorem. Namely the function

$$F(\boldsymbol{x}, t) = [\boldsymbol{\Phi}_t(\boldsymbol{x}) - \boldsymbol{x}_\star] \cdot \boldsymbol{f}(\boldsymbol{x}_\star) \tag{2.18}$$

satisfies

$$F(\boldsymbol{x}_\star, T) = 0 \tag{2.19}$$

and

$$(\partial_t F)(\boldsymbol{x}_\star, T) = \dot{\boldsymbol{\Phi}}_t(\boldsymbol{x}_\star) \cdot \boldsymbol{f}(\boldsymbol{x}_\star) = \parallel \boldsymbol{f} \parallel^2 (\boldsymbol{x}_\star) > 0 \tag{2.20}$$

Hence by implicit function theorem there must exist a ball $\mathrm{B}_\delta(\boldsymbol{x}_\star)$ where

$$F(\boldsymbol{x}, \tau \circ \boldsymbol{x}) = 0 \tag{2.21}$$

or equivalently

$$0 = \mathrm{d}F(\boldsymbol{x}_\star, T) = \mathrm{d}\boldsymbol{x} \cdot (\partial_{\boldsymbol{x}} F)(\boldsymbol{x}_\star, T) + \mathrm{d}t \;\parallel \boldsymbol{f} \parallel^2 (\boldsymbol{x}_\star) \tag{2.22}$$

the solution whereof for $t \sim T$ specifies the function τ. □

We can therefore always hypothesize

$$\mathrm{S} = \mathrm{H} \cap \mathrm{B}_\delta(\boldsymbol{x}_\star) \tag{2.23}$$

so that

$$\frac{(\boldsymbol{n} \cdot \boldsymbol{f})(\boldsymbol{x}_\star)}{(\parallel \boldsymbol{n} \parallel \parallel \boldsymbol{f} \parallel)(\boldsymbol{x}_\star)} = 1 \tag{2.24}$$

Definition 2.3. *We call Poincaré map the application*

$$\boldsymbol{P}\colon \{\boldsymbol{x} \in \mathrm{S} \,|\, \tau(\boldsymbol{x}) < \infty\} \mapsto \mathrm{S} \tag{2.25}$$

defined by the relation

$$\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{\Phi}_{\tau(\boldsymbol{x})}(\boldsymbol{x}) \tag{2.26}$$

Let us consider the linearization of the Poincaré in around $\boldsymbol{x}_\star \in \mathrm{S} \cap \mathrm{O}$

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{x}) &= \boldsymbol{P}(\boldsymbol{x}_\star) + (\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) \cdot (\boldsymbol{x} - \boldsymbol{x}_\star) + \text{h.o.t.} \\ &= \boldsymbol{x}_\star + (\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) \cdot (\boldsymbol{x} - \boldsymbol{x}_\star) + \text{h.o.t.}\end{aligned} \tag{2.27}$$

We can now relate $(\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star)$ to the monodromy matrix

$$(\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) = \boldsymbol{f}(\boldsymbol{x}_\star) \otimes (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) + \mathsf{M} \tag{2.28}$$

or alternatively

$$\mathsf{M} = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) - \boldsymbol{f}(\boldsymbol{x}_\star) \otimes (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) \tag{2.29}$$

We know already that

$$\mathsf{M} \cdot \boldsymbol{f}(\boldsymbol{x}_\star) = \boldsymbol{f}(\boldsymbol{x}_\star) \tag{2.30}$$

It is therefore expedient to introduce a frame of reference such that

$$\boldsymbol{e}_1 = \frac{\boldsymbol{f}(\boldsymbol{x}_\star)}{\parallel \boldsymbol{f}(\boldsymbol{x}_\star) \parallel} \qquad \& \qquad \boldsymbol{e}_i \in \mathrm{S} \quad \forall\, i = 2, \dots, d \tag{2.31}$$

Furthermore if we choose S such that (2.24) holds we also have

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_i = 0 \qquad \forall\, i = 2, \dots, d \tag{2.32}$$

and by hypothesis

$$P^1(\boldsymbol{x}) := \boldsymbol{e}_1 \cdot \boldsymbol{P}(\boldsymbol{x}) = 0 \tag{2.33}$$

since by orthogonality any point on S has no projection over O. Hence

$$\begin{aligned}0 = \boldsymbol{e}_i \cdot \partial_{\boldsymbol{x}} P^1(\boldsymbol{x}) &= \boldsymbol{e}_1 \cdot [\boldsymbol{f}(\boldsymbol{x}_\star) \otimes (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) + \mathsf{M}] \cdot \boldsymbol{e}_i \\ &= \parallel \boldsymbol{f}(\boldsymbol{x}_\star) \parallel \, \boldsymbol{e}_i \cdot (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) + \boldsymbol{e}_1 \cdot \mathsf{M} \cdot \boldsymbol{e}_i\end{aligned} \tag{2.34}$$

which contrasted with (2.30) yields

$$\boldsymbol{e}_1 \cdot (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) = -\frac{1}{\parallel \boldsymbol{f} \parallel (\boldsymbol{x}_\star)} \tag{2.35}$$

More generally, the monodromy matrix takes the form

$$\mathsf{M} = \begin{bmatrix} 1 & -\parallel \boldsymbol{f} \parallel (\boldsymbol{x}_\star)\, \boldsymbol{e}_2 \cdot (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) & \dots & -\parallel \boldsymbol{f} \parallel (\boldsymbol{x}_\star)\, \boldsymbol{e}_d \cdot (\partial_{\boldsymbol{x}}\tau)(\boldsymbol{x}_\star) \\ 0 & & & \\ \vdots & & (\partial_{\boldsymbol{x}_\perp} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) & \\ 0 & & & \end{bmatrix} \tag{2.36}$$

where $(\partial_{\boldsymbol{x}_\perp} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) \in \mathbb{R}^{(d-1)\times(d-1)}$ stands for the restriction of $(\partial_{\boldsymbol{x}} \otimes \boldsymbol{P})(\boldsymbol{x}_\star)$ to S. As the first column contains only one non-vanishing entry the characteristic polynomial factorizes into the product

$$\det(\mathsf{M} - m\,\mathbb{1}) = (1 - m)\, \det(\partial_{\boldsymbol{x}_\perp} \otimes \boldsymbol{P} - m\,\mathbb{1}) \tag{2.37}$$

The conclusion is that the spectrum of the monodromy matrix satisfies

$$\mathrm{Sp}\,\mathsf{M} = 1 \oplus \mathrm{Sp}\,(\partial_{\boldsymbol{x}_\perp} \otimes \boldsymbol{P})(\boldsymbol{x}_\star) \tag{2.38}$$

Remark 2.1. The factorization property (2.37) is a consequence of (2.30) alone.

References

[1] N. Berglund. Perturbation theory of dynamical systems, 2001, math/0111178.

[2] L. Perko. *Differential Equations and Dynamical Systems*. Springer, 3rd edition, 2006.