# Billiard tables



Def. Let $D \subset \mathbb{R}^2$ be a domain with smooth or piece-wise smooth boundary. A billiard system corresponds to a free motion of a point particle inside $D$ with specular reflections off the boundary $\partial D$.

→ Specular reflection: angle of incidence is equal to the angle of reflection

→ Special care on the boundary $\partial D$:
does $\partial D$ have - infinite length
- unbounded curvature
- infinitely many inflection points

~~[illegible]~~

(*) One usually assumes that these or other patologies do not occur as these may render the properties of the dynamics barely tractable

Assumptions of regularity

Let $D_0 \subset \mathbb{R}^2$ be a bounded open connected domain and $D = \overline{D_0}$ denote its closure

A1. The boundary $\partial D$ is a finite union of smooth ($C^\ell$, $\ell \geq 3$) compact curves:

$$\partial D = \Gamma = \Gamma_1 \cup \dots \cup \Gamma_r$$

(*) Precisely, each curve $\Gamma_i$ is defined by a $C^\ell$ map $f_i : [a_i, b_i] \to \mathbb{R}^2$ which is one-to-one on $[a_i, b_i)$ and has one-sided derivatives up to order $\ell$ at the points $a_i$ and $b_i$

The value of $\ell$ is the class of smoothness of the billiard

We will refer to $D$ a billiar table and $\Gamma_i$, walls or components of $\partial D$

Remark If $f_i(a_i) \neq f_i(b_i)$ then we call $\Gamma_i$ an arc
On the contrary, ~~Instead~~ if $f_i(a_i) = f_i(b_i)$ then $\Gamma_i$ is a closed curve

A2 The boundary components can intersect each other only at the end points

$$\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cup \partial\Gamma_j \quad \text{for } i \neq j$$

We call $x \in \partial\Gamma_1 \cup \ldots \cup \partial\Gamma_r$ corner points

(*) | if $x$ is not a corner point then one can always find an open neighborhood $U(x)$ ~~such that~~ ~~$U(x) \cap \Gamma$~~ that intersects only one wall $\Gamma_i$ and is divided by $\Gamma_i$ into two parts — one lies in the interior of $D$ and the other in the exterior $\mathbb{R}^2 \setminus D$.

¿(*)? | we will deal with simple corner points

(*) | Note that a corner $x$ is called a cusp when the angle at which the two wall converging on $x$ intersect is zero.

A3. On every $\Gamma_i$, $f_i''$ either never vanishes or is ~~always~~ identically zero

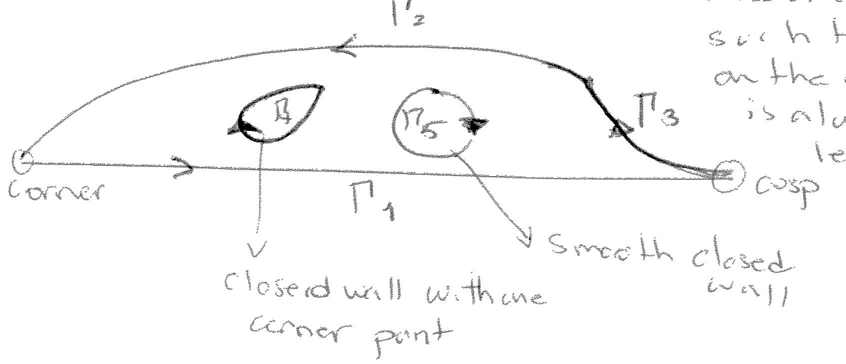
(*) | every wall is a curve without inflection points or ~~a~~ line segment

Three types of walls:

Flat walls: s.t. $f'' \equiv 0$

Focusing walls: s.t. $f'' \neq 0$ is pointing inside $D$

Dispersing walls: s.t. $f'' \neq 0$ is pointing outside $D$




The orientation is such that by moving on the wall ~~they~~ $D$ is always at the left.

$\Gamma_1$ is flat

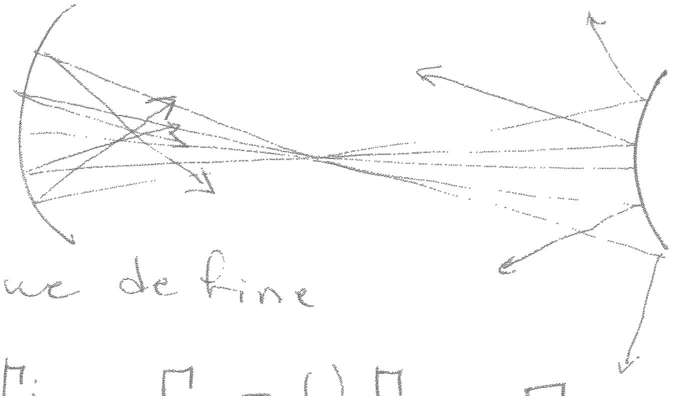
$\Gamma_2$ is focussing

$\Gamma_3$ is dispersing

We convene the following definition

Def. (signed) curvature

$$\kappa = \begin{cases} 0 & \text{if } \Gamma_i \text{ is flat} \\ -\|f_i''\| & \text{if } \Gamma_i \text{ is foccussing} \\ \|f_i''\| & \text{if } \Gamma_i \text{ is dispersing} \end{cases}$$



Accordingly we define

$$\Gamma_0 = \bigcup_{\kappa=0} \Gamma_i \,, \quad \Gamma_- = \bigcup_{\kappa<0} \Gamma_i \,, \quad \Gamma_+ = \bigcup_{\kappa>0} \Gamma_i$$

We convene that each wall has finite length $|\Gamma_i| < \infty$ and set $|\Gamma| = \sum_i |\Gamma_i|$ to be the total perimeter of $D$

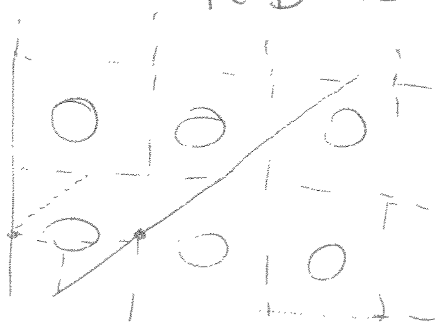
- Unbounded billiard tables

→ we will ask that the boundary locally piece-wise smooth. This is, for any large square $K_B \in \mathbb{R}^2$ of size $B^2$ the intersection $D \cap K_B$ must have a finitely piece-wise smooth boundary satisfying the assumptions A1-A3.

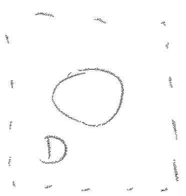
- Unbounded billiard with periodic structure

This happens if there are two orthogonal unit vectors $u, v \in \mathbb{R}^2$ such that

$$q \in D \iff q + u \in D \iff q + v \in D$$



periodic boundary conditions map a periodic unbounded billiard into $Tor^2$




$Tor^2$

obviously this satisfies A1-A3

Summarizing

<u>Def.</u> A billiard table $D$ is the closure of a bounded open connected domain $D \subset \mathbb{R}^2$ or $D \subsetneq Tor^2$ such that $\partial D$ satisfies assumptions A1-A3.

# Billiard in a circle

Let

$$q_t = (x_t, y_t) \quad \text{the position}$$

$$v_t = (u_t, w_t) \quad \text{the velocity}$$

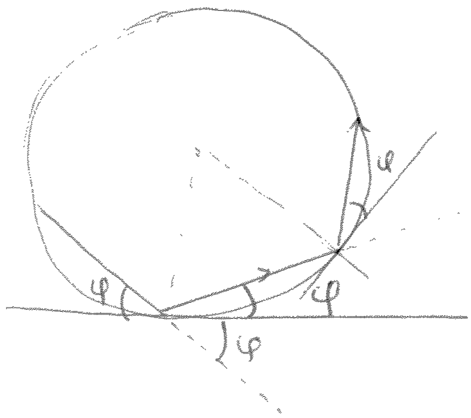
Inside $D$ the unit ~~disc~~ circle

$$x_{t+s} = x_t + u_t s \qquad u_{t+s} = u_t$$

$$y_{t+s} = y_t + w_t s \qquad w_{t+s} = w_t$$

When the particle collides with $\partial D = \{x^2 + y^2 = 1\}$

the velocity vector gets reflected across the tangent line $t\,\partial D$ at the point of collision




$$v^{new} = v^{old} - 2\langle v^{old}, \hat{n}\rangle \hat{n}$$

$$\langle v, n\rangle = ux + wy$$

- The angle of incidence do not change

Let parametrize the disc by the polar (counterclock wise) angle $\theta \in [0, 2\pi]$ and denote by $\varphi$ the angle of reflection

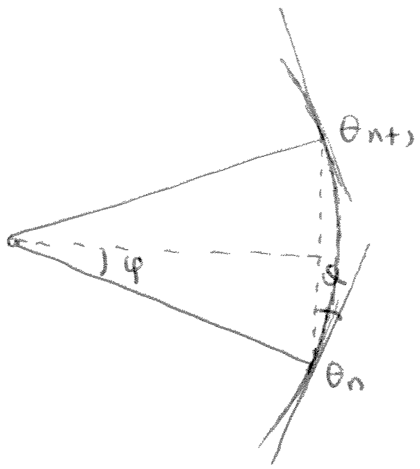
$\theta$ is an arc length (This parametrization will be common in the following lessons)

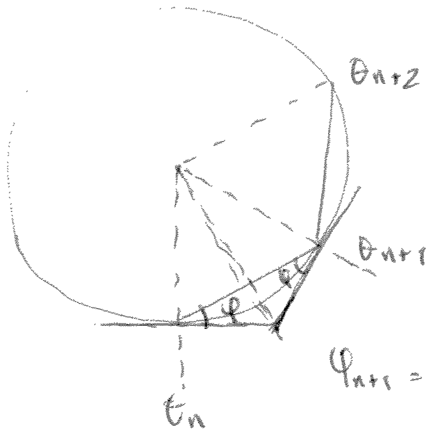
Then

$$\theta_{n+1} = \theta_n + 2\varphi_n \pmod{2\pi}$$

$$\varphi_{n+1} = \varphi_n$$

for all $n \in \mathbb{Z}$




$\varphi_{n+1} = \varphi_n$ by symetry

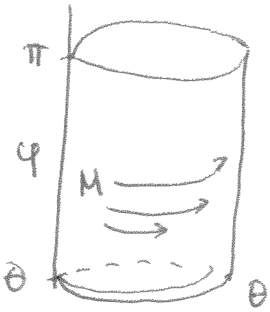
This defines a collisional map $F : M \to M$

Let $(\theta_0, \varphi_0)$ denote the initial collision.
Then

$$\theta_n = \theta_0 + 2n\varphi_0 \pmod{2\pi}$$

$$\varphi_n = \varphi_0$$

The speed in $\theta$ depends on the initial angle $\varphi_0$



Rotations through rational angles are periodic while through irrational ones are ergodic

- If $\frac{\varphi}{\pi} = \frac{m}{n}$ with $m, n$ integers then the trajectory is periodic with period $n$

$$\theta_k = \theta_0 + 2k\varphi_0$$

$$= \theta_0 + \frac{2km\pi}{n}$$

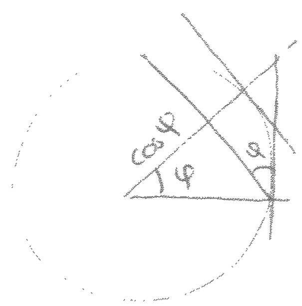
Thus, when $k = n$ $\quad \theta_k = \theta_0 + 2\pi m$

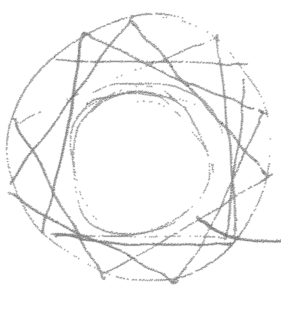
- if $\varphi/\pi$ is irrational then the rotation is ergodic w.r.t. the Lebesgue measure:

  for every point $(\varphi, \theta) \in C_\varphi$ its images $(\varphi, \theta + 2n\varphi)$ are dense and uniformily distributed on $C_\varphi$.

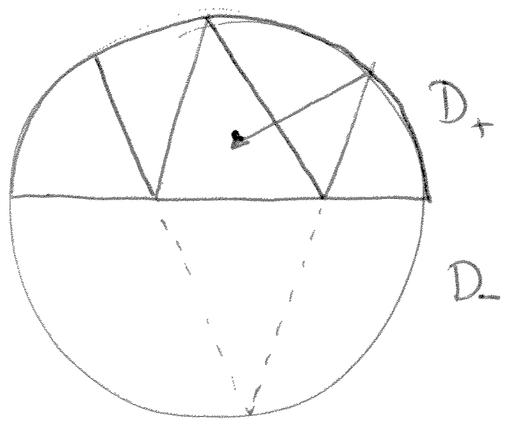
- Every segment of the trajectory bewteen to consecutive collisions is tangent to a smaller circle

$S_\varphi = \{x^2 + y^2 = \cos^2\varphi\}$



the inner circle is called caustic (from the greek burning)




A periodic orbit in $D$ is also periodic in $D_+$ but with a period which is twice the original

Thys type of symmetrization will be used commonly

# SUMMARY 1

Let $D_0 \subset \mathbb{R}^2$ be a <u>bounded</u> <u>open</u> <u>connected</u> domain and $D = \overline{D_0}$ its closure.

A billiard system corresponds to a free motion of a point particle in $\mathrm{int}\, D$ with specular reflections off the boundary $\partial D$

A1. The boundary $\partial D$ is a finite union of smooth $C^\ell$ $(\ell \geq 3)$ compact curves

$$\partial D = \Gamma = \Gamma_1 \cup \dots \cup \Gamma_k$$

$\Gamma_i =:$ $C^\ell$ map $f_i : [a_i, b_i] \to \mathbb{R}^2$

one-to-one on $[a_i, b_i)$

one-sided derivatives of order $\ell$ exists at $a_i, b_i$

$D \leftarrow$ billiard table

$\Gamma_i \leftarrow$ billiard walls

A2. The boundary components can intersect each other only at the end points

$$\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cup \partial\Gamma_j \quad \text{for all } i \neq j$$

$x \in \overbrace{\partial\Gamma_1 \cup \dots \cup \partial\Gamma_k}^{\equiv \Gamma_*} \leftarrow$ corner points $\quad \tilde{\Gamma} = \Gamma \setminus \Gamma_*$

if $x$ ~~$\partial\Gamma_i \cup \partial\Gamma_j$~~ $\Gamma_i \cap \Gamma_j = 0$ $\quad x \in \begin{cases} \Gamma_* & \text{corner points} \\ \tilde{\Gamma} & \text{regular points} \end{cases}$

then $x$ is a cusp

# SUMMARY 2

A3. On every $\Gamma_i$, $f_i''$ either never vanishes or is identically zero.

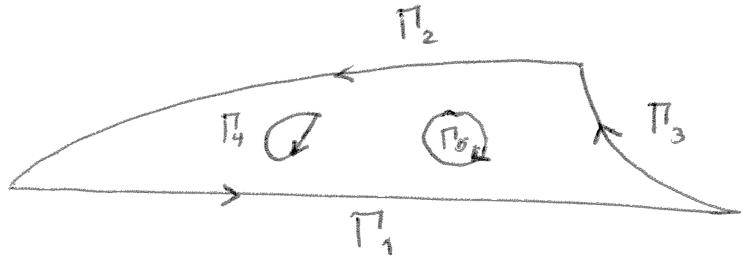
$\boxed{\text{no inflection points}}$

The curvature of $\Gamma$, $f''$ serve us to define the types of walls

Flat walls : s.t. $f''=0$

Focussing walls : s.t. $f''\neq 0$ points inside D

Dispersing walls : s.t. $f''\neq 0$ points outside D




$\Gamma_4$ : closed curve with one corner point

$\Gamma_5$ : smooth closed curve

* orientation : by moving on the wall D is always at the left

# Accomulation of collision times 

<u>Def.</u> The trajectory $(q(t), v(t))$ starting at $q(0) \in \operatorname{int} D$ is defined at all times $-\infty < t < \infty$ unless:

a) the particle hits a corner point, $q(t) \in \Gamma_*$ for some $t \in \mathbb{R}$

b) collision times have an accumulation point in $\mathbb{R}$.

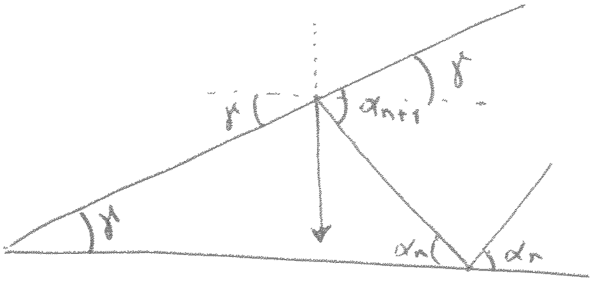
Case (b): $t_n \to t_\infty \in \mathbb{R}$ s.t. $q(t_n) \to q_\infty \in \Gamma$ | Having an infinite number of collisions in a finite time

We have two caseas

$b_1$) $q_\infty \in \Gamma_*$ (a corner point)

$b_2$) $q_\infty \in \Gamma$ (a regular point)

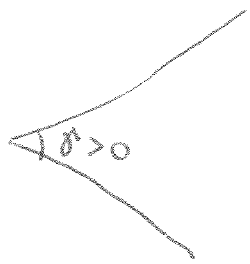
Lemma. Suppose the trajectory enters the region of a corner point wit a positive interior angle $\gamma > 0$ and collides with bothsides. Then, it must leave that ~~regi~~ neighbourhood after at most $[\pi/\gamma] + 1$ collisions so that ($b_1$) cannot occur.




$\alpha_{n+1} = \alpha_n + \gamma$

Thus, after $\left[\frac{\pi}{\gamma}\right]$ collisions

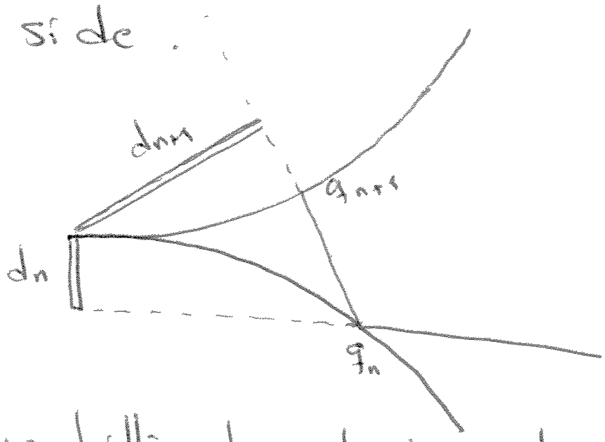
$\alpha_{n+1} > \pi$ ∎




This case follows by simple approximations

This is independent of the trajectory

cusps $\gamma = 0$ must have at least one dispersing side.




Lemma. if the billiard particle enters a cusp with two dispersing sides or one dispersing side and one flat then it must leave after a finite number of collisions, so that $(b_1)$ cannot occur.

One shows this by proving that the sequence of distances $\{d_n\}$ is an increasing sequence for dispersing and constant for flat

! Note, however, that the # of collisions in a cusp is not uniformly bounded :

For any $N \geq 1$ $\exists$ a billiar trajectory that experiences more than $N$ collisions in the vicinity of a cusp before leaving.

This is in contrast with corners with $\gamma > 0$ for which a uniform bound exists.

The only possibility is  but this is an open problem. To exclude this possibility we make the fourth assumption

<u>Assumption 4</u> Any billiard table $D$ contains no cusps made by a foccusing wall and a dispersing wall.

Case (b2) $q_\infty \in \tilde{\Gamma}$

clearly (b2) is not possible in flat or dispersing walls

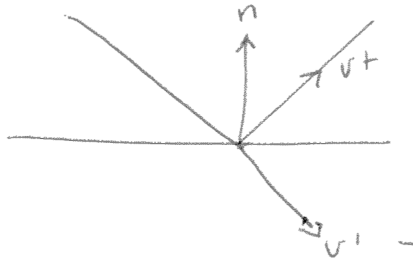
Theorem (B. Halpern) The type (b2) accumulation of collisions is impossible on any foccusing wall with a bounded third derivative and nowhere vanishing curvature.

[Halpern found a b2 accum. point for $C^2$ focusing billiard wall]

## Billiard flow

In the interior — $q(t), v(t)$ are position and velocity 

$$\dot q = v \,, \quad \dot v = 0$$




$$v^+ = v^- - 2\langle v, n\rangle n \quad \text{at collisions}$$

The eq.'s of motion preserves $\|v\|$

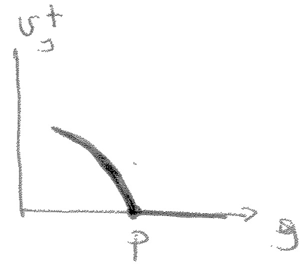
So we set $\|v\| = 1$

→ COMPLICATION:
When a particle hits a corner point, it stops.

Def. A collision is said to be "regular" if $q \in \Gamma$ is not a corner point and $v^-$ is not tangent to $\Gamma$.

If $v^-$ is tangent to $\Gamma$ then $v^+ = v^-$ and the collision is said to be grazing or tangential

(*) Of course it is evident that a tangential collision can occur only on dispersing walls. Also ~~to~~ note that at tangential collisions the flow is not differentiable

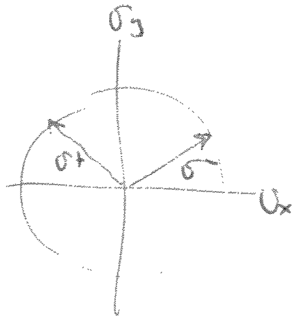
# Phase space for the flow

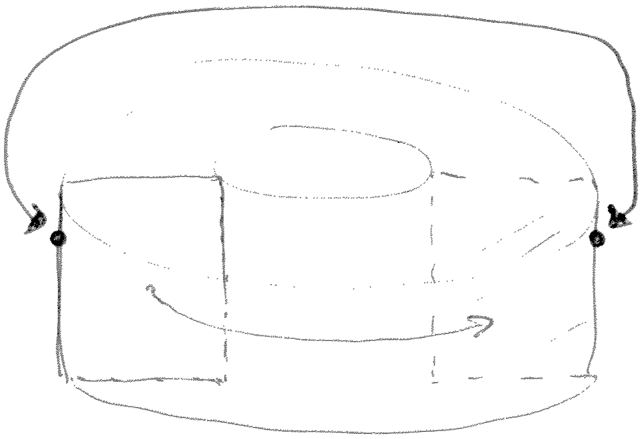
The phase space of the system is

$$\Omega = \{(q, v)\} = D \times S^1$$

$q \in D$ and $v \in S^1$ is the unit circle

$\Omega$ is a 3D manifold with boundary $\partial \Omega = \Gamma \times S^1$



$\Omega$ is a ~~dnought~~ doughnut with cross-section D.

At each regular boundary points $q \in \tilde{\Gamma}$ we identify the pairs $(q, v^-)$ and $(q, v^+)$

where $v^+ = v^- - 2\langle v, n\rangle n$

which amounts of gluing $\Omega$ along its boundary.

Denote $\Pi_q$ and $\Pi_v$ the natural projections of $\Omega$ on D and $S^1$

$$\Pi_q(q, v) = q$$

$$\Pi_v(q, v) = v$$

Let $\tilde{\Omega} \subset \Omega$ denote the set of states $(q, v)$ on which the dynamics is defined at all times

<u>Def</u> The billiard flow is thus defined as a one-parameter group

$$\Phi^t : \tilde{\Omega} \longrightarrow \tilde{\Omega}$$

with continuous time $t \in \mathbb{R}$

$$\Phi^0 = \mathrm{Id}$$

$$\Phi^{t+s} = \Phi^t \circ \Phi^s \quad \forall \; t, s \in \mathbb{R}$$

Every trajectory of the flow $\{\Phi^t x\}$, $x \in \tilde{\Omega}$ is a continuous curve in $\Omega$ (by construction, after the identification of $v^+$ and $v^-$.

→ Note a billiard trajectory is $\pi_q(\phi^t x)$

For bounded billiards the trajectory segments cannot be larger than the diameter of D.

Lemma: if D is a billiard table in $\mathrm{Tor}^2$ then every trajectory of the flow experiences either infinitely many collisions or none at all

proof → noting that in $\mathrm{Tor}^2$ every free-trajectory is either periodic or dense

The important thing is the following

Let $\tilde{\Omega} = \tilde{\Omega}_c \cup \tilde{\Omega}_f$

$\tilde{\Omega}_c$ = set of all trajectories with collisions

$\tilde{\Omega}_f$ = union of all collision-free trajectories which may exists only if $D \subsetneq \mathrm{Tor}^2$ like in

Then both $\tilde{\Omega}_c$ and $\tilde{\Omega}_f$ are invariant under $\Phi^t$

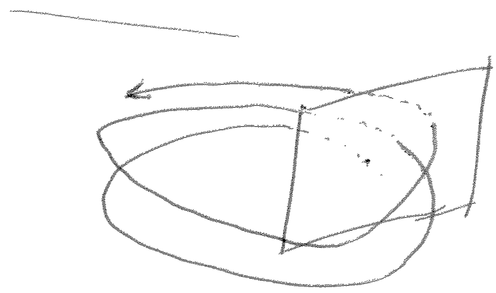
In the example above, there are only 2 velocities $v \in S^1$ corresponding to points $x \in \tilde{\Omega}_f$

<u>Def.</u> We say that a billiard table has finite horizon if $\tilde{\Omega}_f = \phi$. Otherwise the horizon is said to be infinite.

$\rightarrow$ [ For finite horizon billiards the segment trajectory between collisions is bounded above by some constant $\tau_{max}(D) < \infty$

## Collision map or Billiard map

Given a ~~[illegible]~~ flow $\Phi^t : \Omega \to \Omega$ on a manifold $\Omega$ one finds a hypersurface $M \subset \Omega$ transversal to the flow so that each trajectory intersects $M$ infinitely many times.



Then the flow induces a "return map"

$F : M \to M$ and a return time function on $M$

$L(x) = \min\{ s > 0 : \Phi^s(x) \in M \}$, so that

$$\boxed{F(x) = \Phi^{L(x)}(x)}$$

$L(x) : M \to \mathbb{R}_+$

conversely: $M$ a measurable space, $F : M \to M$ a measurable map
$L : M \to \mathbb{R}_+$ a positive function.
One can construct the space

$$\Omega = \{(x,s) : x \in M,\ 0 \leq s \leq L(x)\}$$

and flow $\Phi^t : \Omega \to \Omega$ defined by $\Phi^t(x,s) = (x, s+t)$ with the identification $(x, L(x))$ and $(F(x), 0)$

→ The flow $\Phi^t$ is measurable on $\Omega$

→ If $F$ preserves a prob. measure $\mu_{\ast}$ on $M$ and $\int d\mu(x) = \int d\mu(F(x))$
$\bar{L} = \int_M L(x)\, d\mu(x) < \infty$

then the flow preserves the prob. measure $\mu_1$ on $\Omega$ defined by

$$d\mu_1 = \bar{L}^{-1} d\mu \times ds$$

this is locally a product measure whose invariance under $\Phi^t$ is a consequence of the Fubini theorem.

~~The map~~ Then we call

$F$ base transformation

$L(x)$ ceiling function

$\Phi^t$ suspension flow

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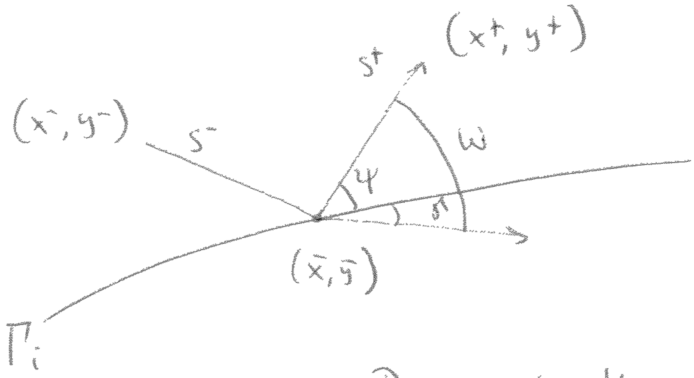
Next class we will introduce a coordinate system, compute the Jacobean of the flow and the derivative of the map.

Time reversal $\neq$ involution.

# Coordinate representation of the flow






Describe the flow $\Phi^t$ in coordinates $(x, y, \omega)$ as

$$q = (x, y) \in D$$

$\omega$ = $\measuredangle$ between $v$ and the positive x axis

$$\omega \in [0, 2\pi)$$

$$(x^-, y^-, \omega^-) \xrightarrow{\Phi^t} (x^+, y^+, \omega^+)$$

we want to compute the derivative of the map

without collisions

$$x^+ = x^- + t \cos\omega$$

$$y^+ = y^- + t \sin\omega$$

$$\omega^+ = \omega^-$$

Suppose there is exactly one (regular collision at $(\bar{x}, \bar{y}) \in \Gamma_i$ during the interval $(0, t)$

Let call $\gamma$ the ∡ between the tangent at $(\bar{x},\bar{y})$ and the positive x-axis. (2)

intermediate variables

- $\psi$ the angle between $v^+$ and the tangent
- $s^-$ the collision time
- $s^+ = t - s^-$

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$$x^- = \bar{x} - s^- \cos\omega^- \qquad x^+ = \bar{x} + s^+ \cos\omega^+$$

$$y^- = \bar{y} - s^- \sin\omega^- \qquad y^+ = \bar{y} + s^+ \sin\omega^+$$

$$\omega^- = \gamma - \psi \qquad \omega^+ = \gamma + \psi$$

Let $r$ denote the arc length parameter on $\Pi_i$. $r \in [a_i, b_i]$

Then

$$d\bar{x} = \cos\gamma \, dr$$

$$d\bar{y} = \sin\gamma \, dr$$

$$d\gamma = -K \, dr \quad , \quad K \text{ is the curvature}$$

the sign of $d\gamma$ follows from our definition of orientation

Differentiating we obtain

$$\begin{pmatrix} dx^+ \\ dy^+ \\ dw^+ \end{pmatrix} = \begin{pmatrix} \cos\gamma\, dr & \cos w^+ ds^+ & -s^+ \sin w^+ dw^+ \\ \sin\gamma\, dr & \sin w^+ ds^+ & s^+ \cos w^+ dw^+ \\ -k\, dr + d\psi & & \end{pmatrix}$$

$$dx^- = \cos\gamma\, dr - \cos w^- ds^- + s^- \sin w^- dw^-$$
$$dy^- = \sin\gamma\, dr - \sin w^- ds^- - s^- \cos w^- dw^-$$
$$dw^- = -k\, dr - d\psi$$

One can show

$$dx^+ \wedge dy^+ \wedge dw^+ = \sin\psi\, dr \wedge ds^+ \wedge d\psi$$

$$dx^- \wedge dy^- \wedge dw^- = -\sin\psi\, dr \wedge ds^- \wedge d\psi$$

The Jacobian $(r, s, \psi) \mapsto (x^\pm, y^\pm, w^\pm)$

equals $\pm \sin\psi$

Note that $s^- + s^+ = t = \text{const}$

Thus

$$ds^+ + ds^- = 0$$

Meaning $dx^+ \wedge dy^+ \wedge dw^+ - dx^- \wedge dy^- \wedge dw^- = 0$

Doing an induction on the number of collisions implies

<u>Theorem</u> The flow $\Phi^t$ preserves the volume form $dx \wedge dy \wedge d\omega$; thus it preserves the Lebesgue measure $dx\,dy\,d\omega$ on $\Omega$

<u>Remark</u> On $\Omega$ the Lebesgue measure is the Liouville measure corresponding to a Hamiltonian system.

Although is difficult to prove that Hamiltonian character is preserved under collisions

Thus we define

<u>Def</u> The normalized Lebesgue measure on $\Omega$

$$d\mu_\Omega = \frac{1}{2\pi |D|}\, dx\, dy\, d\omega$$

is the canonical probability measure preserved by the billiard flow $\Phi^t$

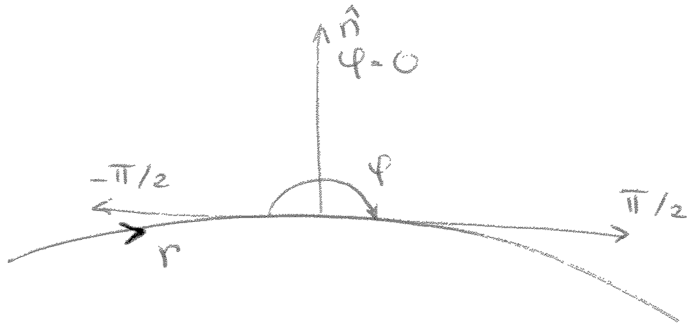
$|D|$ is the area of the billiard.

# Coordinates for the collision map

Let $r$ ~~be~~ denote the arclength on each $\Gamma_i$. $r : [a_i, b_i] \to \mathbb{R}$

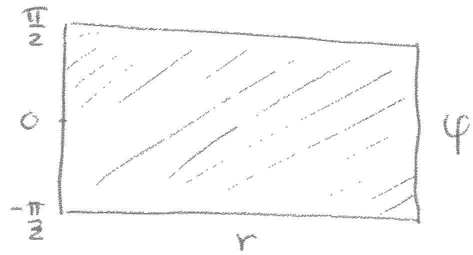
We recall $b_i - a_i = |\Gamma_i|$ and that the intervals $(a_i, b_i)$ are disjoint in $\mathbb{R}$

If $\Gamma_i$ is a smooth closed curve $(a_i, b_i)$ then $r$ is a cyclic parameter.




Let $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote the angle between $v$ and $n$ oriented as in the figure.

Then $r$ and $\varphi$ make coordinates on $M$




For each wall $\Gamma_i$, the manifold $M_i$ is

$$M_i = \begin{cases} \underbrace{\Gamma_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]}_{\text{cylinder}} & \text{for closed curves} \\ [a_i, b_i] \times [-\frac{\pi}{2}, \frac{\pi}{2}] & \text{for regular walls.} \end{cases}$$

Let us define the boundary of $M$

$$S_0 := \partial M = \{|\varphi| = \tfrac{\pi}{2}\} \cup \underbrace{\left( \bigcup_i \left( \{r = a_i\} \cup \{r = b_i\} \right) \right)}_{\text{for non closed curves}}$$

For every point $x \in \operatorname{int} M$ its trajectory $\Phi^t x$ is defined at least until the next intersection with $M$ $0 < t < \tau(x)$, at which we have three posibilities:

(a) a regular collision, $F(x) \in \operatorname{int} M$

(b) a grazing collision, $F(x) \in S_0$

(c) the trajectory hits a corner point and dies

Denoting $S_1 = S_0 \cup \{ \underline{F(x) \notin \operatorname{int} M} : \underline{x \in \operatorname{int} M} \}$

$S_1 \setminus S_0$ is the set of points in which (b) or (c) occur

$F$ is a local homeomorphism in $M \setminus S_1$

Also, at every point $x \in S_1 \setminus S_0$ the map $F$ is discontinuous

since $F^{-1}$ is also continuous on $M \setminus S_{-1}$ with

$$S_1 = S_0 \cup \{ x \in \operatorname{int} M : F^{-1}(x) \notin \operatorname{int} M \}$$

$F : M \setminus S_1 \to M \setminus S_{-1}$ is a homeomorphism

$$\boxed{\begin{aligned} S_1 &= S_0 \cup F^{-1}(S_0) \\ S_{-1} &= S_0 \cup F(S_0) \end{aligned}}$$

$F^{-1}(S_0)$ : set of points with image in $S_0$

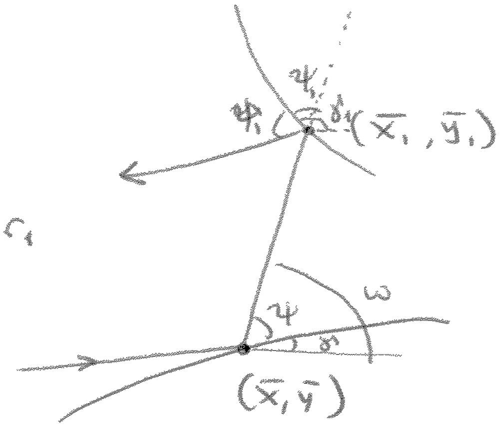
$F(S_0)$ set of points with preimage in $S_0$

## Derivative of the map

Here we differentrate $F$ at a point $x=(r,\varphi)\in \operatorname{int} M$ such that $F(x)=(r_1,\varphi_1)\in \operatorname{int} M$.

$(\bar{x},\bar{y})\in\partial D$ are the coordinates corresponding to $r$

$(\bar{x}_1,\bar{y}_1)\in\partial D$ those corresponding to $r_1$




- $\omega$ is the angle between $\underbrace{(\bar{x},\bar{y}) \text{ and } (\bar{x}_1,\bar{y}_1)}_{\text{segment}}$ and the positive x-axis
- $\psi=\frac{\pi}{2}-\varphi$

Then we have

$$\circledast \qquad \begin{aligned} \bar{X}_1-\bar{X} &= \tau\cos\omega \\ \bar{Y}_1-\bar{Y} &= \tau\sin\omega \end{aligned} \qquad ; \quad \tau=\tau(x)$$

Recall

$$\begin{aligned} d\bar{x} &= \cos\gamma\, dr \\ d\bar{y} &= \sin\gamma\, dr \\ d\gamma &= -K dr \end{aligned}$$

Note also that $\omega:=\gamma+\psi=\gamma_1-\psi_1$

differentrating the last eq gives

$$d\omega=-Kdr+d\psi=-K_1dr_1-d\psi_1$$

differentrating $\circledast$

$$-\cos\varphi_1\, dr_1=(\tau K+\cos\varphi)dr+\tau d\varphi$$

$$-\cos\varphi_1\, d\varphi_1=(\tau KK_1+K\cos\varphi_1+K_1\cos\varphi)dr+(\tau K_1+\cos\varphi_1)d\varphi$$

Thus the derivative $DF$ at point $x = (r, \varphi)$ is

$$DF = -\frac{1}{\cos\varphi_1}\begin{pmatrix} \tau K + \cos\varphi & \tau \\ \tau K K_1 + K\cos\varphi_1 + K_1\cos\varphi & \tau K_1 + \cos\varphi_1 \end{pmatrix}$$

and

$$\det |DF| = \cos\varphi$$

Theorem. The map $F : M \setminus S_1 \to M \setminus S_{-1}$ is a $C^{\ell-1}$ diffeomorphism.

proof The derivative $DF$ is expressed through the curvature $K$ and $K_1$ of the boundary $\partial D$, which corresponds to the second derivative of the $C^\ell$ functions $f_i : [a_i, b_i] \to \mathbb{R}^2$ ∎

→ Note that the derivative is unbounded when $\cos\varphi_1 \to 0$ (when $x_1$ is near $S_0$ and $x$ is near $S_1$)

Clearly $S_1$ is the singularity set of $F$.

Similarly $S_{-1}$ " " " " " $F^{-1}$

By induction

$$S_{n+1} = S_n \cup F^{-1}(S_n)$$

$$S_{-(n+1)} = S_{-n} \cup F(S_{-n})$$

are the singularity sets of $F^{n+1}$ and $F^{-(n+1)}$ respectively

Thus, on the set $\hat{M} := M \setminus \bigcup_{n=-\infty}^{\infty} S_n$

all the iterations of $F$ are defined and are $C^{\ell-1}$ differentiable

## Invariant measure of the map

**Lemma.** The map $F$ preserves the measure $\cos\varphi \, dr \, d\varphi$ on $M$.

Proof: For any Borel set $A \subset M$

$$\iint_{F(A)} \cos\varphi_1 \, dr_1 \, d\varphi_1 = \iint_{A} \cos\varphi \, dr \, d\varphi \qquad \blacksquare$$

Moreover

$$\iint_{M} \cos\varphi \, dr \, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_{\Gamma} dr = 2|\Gamma|$$

two times the perimeter.

**Definition** The normalized measure on $M$

$$d\mu = \frac{1}{2|\Gamma|} \cos\varphi \, dr \, d\varphi$$

is the canonical probability measure preserved by the billiard map $F$.

Recalling that the flow preserved measure is

$$d\mu_1 = \bar{\tau}^{-1} d\mu \times ds$$

$$= (2|\Gamma|)^{-1} \bar{\tau}^{-1} \cos\varphi \, dr \, d\varphi \, ds$$

where

$$\bar{\tau} = \int_M \tau(x) \, d\mu(x)$$ is the mean return time.

Recall that time = distance $^{-1}$
since $\|\omega\| = 1$

Using $dx \wedge dy \wedge d\omega = \sin\psi \, dr \, ds \, d\varphi$

and $\sin\psi = \cos\varphi$

$$d\mu_1 = (2|\Gamma|)^{-1} \bar{\tau}^{-1} dx \, dy \, d\omega$$

Comparing with the invariant measure of the flow we conclude that

$$2\pi |D| = 2|\Gamma| \bar{\tau}$$