Lecture 09: Lyapunov Exponents

1 Introduction

We start this lecture by asking the question: What is chaoticity? Chaos refers to a complicated dynamical behaviour. In the early times of dynamical system theory, the "chaoticity" was customarily related with a *strong dependence on the initial conditions*.

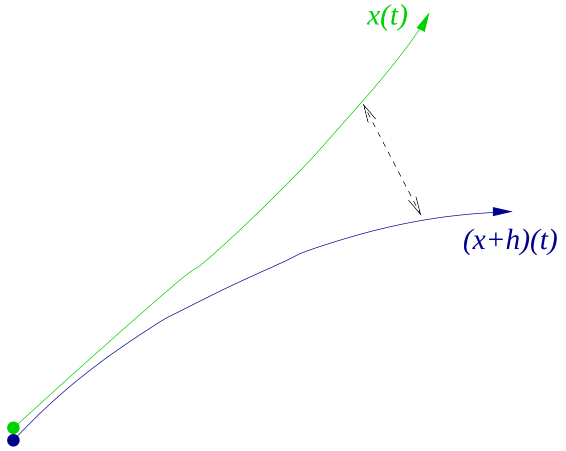


Figure 1: Exponential separation of nearby trajectories.

In this lecture we will introduce the Lyapunov *characteristic* exponents, which characterise quantitatively the chaoticity of a given dynamical system.

2 Preliminaries

2.1 Invariant Measures

Definition 2.1. *Let f be a measurable map $M \to M$. A measure μ is called invariant for f if*

$$\mu(f^{-1}(E)) = \mu(E) \tag{2.1}$$

for every measurable set E.

An important class of measures are those that are Lebesgue-integrable

Definition 2.2. *A measure μ is called absolutely continuous (with respect to the Lebesgue measure) if*

$$\mu(dx) = h(x)dx \quad \text{with} \qquad h \in L^1 \ ; \tag{2.2}$$

that is, $\int h(x)dx < \infty$.

With the help of the ergodic theorem (see next section), which says that time averages of observables are equal to space averages, the invariant measures of a dynamical system can be empirically constructed by density histograms.

Definition 2.3. *Let A be a subset of the phase space Ω and f a measurable map. The average time the "system state" spends in A over a time interval $[0, N]$ starting from the initial "state" x_0 is*

$$\mathcal{A}_N(x_0, A) = \frac{1}{N} \sum_{j=0}^{N-1} \chi_A\left(f^j(x_0)\right) \ , \tag{2.3}$$

where χ_A denotes the characteristic function on the set A.

One is interested in the large-N limit of this quantity. Assuming that the limit exists, and denoting it by $\mu_{x_0}(A)$ then it is easy to check that:

1. the limit also exists for $f(x_0)$ and for any $y \in f^{-1}(x_0) = \{z | f(z) = x_0\}$.
2. the limit also exists if A is replaced by $f^{-1}(A)$ and has the same value, namely

$$\mu_{x_0}(f^{-1}(A)) = \mu_{x_0}(A) \ ; \tag{2.4}$$

   this is indeed true since $\chi_A\left(f^j(x)\right) = 1$ if and only if $x \in f^{-j}(x_0)$.

Remark 2.1. *If one assumes that $\mu_{x_0}(A)$ does not depend on x_0 then, in the limit of large-N the histogram $\mathcal{A}_N(x_0, A)$ resembles an invariant measure.*

Definition 2.4.

$$\tag{2.5}$$

2.2 The ergodic theorem

In the previous section, the histogram (2.3) is a time average of the characteristic funcion of the bin of measure A, which means that one counts how often the orbit visits a given interval I. Thus, the observable ($h(x_0)$) inside the sum of (2.3 takes value 1 if $x \in I$ and 0 otherwise.

➨ The *ergodic theorem* tells us that this procedure converges to the invariant measure for almost every initial condition x_0.

Theorem 2.2. Birkhoff's ergodic theorem *Let μ be an invariant probability measure for the map f (on the space X). Let h be an integrable function on X: $\int_X d\mu < \infty$. Define the partial sums:*

$$S_n(x) = \sum_{i=0}^{n-1} h(f^i(x)) \ . \tag{2.6}$$

1. *For μ-almost every x one has*
$$\lim_{n\to\infty}\frac{1}{n}S_n(x) = h^*(x)\,. \tag{2.7}$$
2. *The function h^* is f-invariant*
$$h^*(x) = h^*(f(x))\,. \tag{2.8}$$
3. *One has*
$$\int h^*(x)d\mu = \int h(x)d\mu\,. \tag{2.9}$$

➡ The notion of μ-almost every x means for a set of x of μ-measure 1.

➡ The space X with measure μ is called a *probability space* if $\int_X d\mu = 1$. In this case μ is called a *probability measure*.

If the limit in the ergodic theorem is independent of the initial condition then the measure μ is ergodic.1

Definition 2.5. *Given a map f on a space X, an invariant probability measure μ is called ergodic with respect to f if $f^{-1}(E) = E$ implies $\mu(E) = 1$ or $\mu(E) = 0$. In other words, the only invariant sets E are those of measure 0 or 1, and in particular, it is not possible that a part of the space is invariant.*

The consequences of having an ergodic measure μ can be stated in the following

Lemma 2.3. *Consider a dynamical system (f, X) with invariant measure μ.*

i) *If μ is ergodic, then for any integrable function h the limit function given by the ergodic theorem is almost surely constant, and furthermore*
$$h(x) = \lim_{N\to\infty}\frac{1}{N+1}\sum_{j=0}^{N} h(f^j(x)) = \int h(y)d\mu(y)\,. \tag{2.10}$$

ii) *If, for all integrable h, the function $h(x)$ is almost surely constant, then μ is ergodic.*

iii) *If μ is ergodic, then any invariant function is μ-almost surely constant.*

iv) *If every invariant function is almost surely constant, then μ is ergodic.*

3 Lyapunov exponents

We discuss the concept of Lyapunov exponents for discrete time maps. We consider a differentiable map f in Ω. For simplicity we will fix $\Omega = \mathbb{R}^d$, the d-dimensional Euclidean space.

Let $\mathbf{h}$ be a vector in Ω such that $|\mathbf{h}| \ll 1$, and consider two nearby trajectories with initial conditions $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$. Our aim is to estimate the behaviour of the initial error $\mathbf{h}$ in time. At time n, the magnitude of the initial error is
$$|f^n(\mathbf{x}+\mathbf{h}) - f^n(\mathbf{x})|\,, \tag{3.1}$$

where $f^n(\mathbf{x}) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(\mathbf{x})$. As long as the error does not grow too much, we can expect to have a reasonable estimate of the error at time n from the first-order Taylor expansion

$$f^n(\mathbf{x}+\mathbf{h}) - f^n(\mathbf{x}) = \mathcal{D}_{\mathbf{x}} f^n \mathbf{h} + \mathcal{O}(\mathbf{h}^2)\,, \tag{3.2}$$

where $\mathcal{D}_{\mathbf{x}} f^n$ denotes the differential of the map f^n at point $\mathbf{x}$. Applying the chain-rule one finds

$$\mathcal{D}_{\mathbf{x}} f^n = \mathcal{D}_{f^{n-1}(\mathbf{x})} f \cdot \mathcal{D}_{f^{n-2}(\mathbf{x})} f \cdot \ldots \mathcal{D}_{\mathbf{x}} f\,. \tag{3.3}$$

One has to be careful since in general, the matrices in this product do not commute. Let's first take the simple one-dimensional example. In this case the differentials (3.3) commute and

$$f^{n\prime}(x) = \prod_{j=0}^{n-1} f'\left(f^j(x)\right)\,. \tag{3.4}$$

We then look for the exponential growth rate, namely

$$\frac{1}{n}\log\left|f^{n\prime}(x)\right| = \frac{1}{n}\sum_{j=0}^{n-1}\log\left|f'\left(f^j(x)\right)\right|\,. \tag{3.5}$$

Finally, assuming that $\log|f'|$ is for μ an ergodic invariant measure, we can use the ergodic theorem (2.2) to conclude that on a set of full μ-measure, the temporal average converges, and moreover

$$\lambda(x) \equiv \lim_{n\to n}\frac{1}{n}\log\left|f^{n\prime}(x)\right| = \int \log\left|f'\right| d\mu\,. \tag{3.6}$$

The limit λ is calle the Lyapunov exponent of the map f for the measure μ at point x.

➡ The Lyapunov exponent $\lambda(x)$ measures the rate of separation of infinitesimally close initial points arounbd x.

4 Existence of Lyapunov Exponents in higher dimensions

As mentioned before, the existence of the Lyapunov exponents in higher dimensions is far more difficult since the differential (3.3) is in general a product of noncommutative terms. As in the one-dimensional case, if for an ergodic invariant measure μ and a map f the limit

$$\lim_{n\to n}\frac{1}{n}\log||D_x f^n|| \tag{4.1}$$

exists μ-almost surely then, if this quantity is positive some initially small errors are exponentially amplified while if it is negative or zero then initial errors cannot grow very rapidly.

We note that in general, for a fixed vector $\mathbf{h}$

$$||D_x f^n \mathbf{h}||^2 = \langle \mathbf{h}(D_x f^n)^t D_x f^n \mathbf{h}\rangle\,, \tag{4.2}$$

where $\langle\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$ and A^t is the transpose of the matrix A. The studied of the exponential growth of the matrix in the r.h.s. of (4.2) led Oseledec to formulate the following

Theorem 4.1. Oseledec's theorem

Let μ be an ergodic invariant measure for a diffeomorphism f of a compact manifold Ω. Then for μ-almost every initial condition x, the sequence of symmetric nonnegative matrices

$$\left((D_x f^n)^t D_x f^n\right)^{1/2n} ,$$

converges to a symmetric nonnegative matrix Λ (independent of x). Denote by $\lambda_0 > \lambda_1 > \ldots > \lambda_k$ the strictly decreasing sequence of the logarithms of the eigenvalues of the matrix Λ (some of them may have nontrivial multiplicity).

These numbers are called the Lyapunov exponents of the map f for the ergodic invariant measure μ. Moreover, for μ-almost every point x there is a decreasing sequence of subspaces

$$\Omega = E_0(x) \supsetneq E_1(x) \supsetneq \cdots \supsetneq E_k(x) \supsetneq E_{k+1}(x) = \{0\} ,$$

satisfying (μ-almost surely)

$$D_x f E_j(x) = E_j(f(x))$$

and for any $j \in \{0, \ldots, k\}$ and any $\mathbf{h} \in E_j(x)\ E_{j+1}(x)$ one has

$$\lim_{n \to n} \frac{1}{n} \log ||D_x f^n \mathbf{h}|| = \lambda_j$$

➡ Note that the theorem says that

$$||D_x f^n \mathbf{h}|| \sim e^{n\lambda_j} .$$

There may be large or small (subexponential) prefactors which can depend on $\mathbf{h}$ and x.

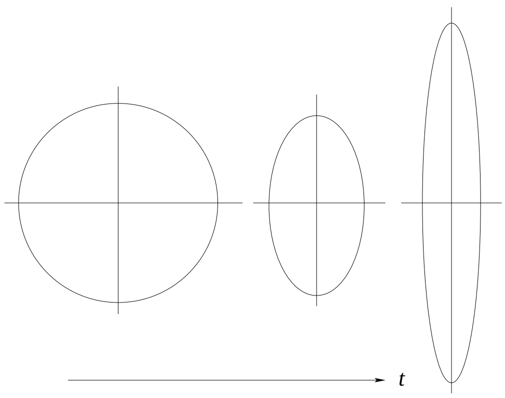


Figure 2: Illustration of the streching and contracting directions.

➡ Positive Lyapunov exponents are obviously responsible for sensitive dependence on initial conditions. Their corresponding "eigen" directions are tangent to the "attractor".

➨ Transversally to the attractor one gets contracting directions, namely negative Lyapunov exponents.

➨ If the map f is area-preserving

$$||D_x f^n|| = \mathbf{1} \text{for all } n \;, .$$

Therefore we have that

$$\sum_{j=0}^{k} \lambda_j = 0 \; .$$

In words, “chaotic dynamics of Hamiltonian systems will strecht(contract) volumes along the tangent(perpendicular) direction of the invariant measure set”.

➨ If the map f is dissipative, meaning that

$$||D_x f^n|| < \mathbf{1} \text{for all } n \;,$$

the Lyapunov exponents will satisfy

$$\sum_{j=0}^{k} \lambda_j < 0 \; .$$

Note that this does not mean that all Lyapunov exponents are negative.

4.1 Iterative calculation of the largest Lyapunov exponents

Numerical determination | For nonlinear systems the Lyapunov exponents are difficult to compute in general |

$$\lim_{n\to\infty} \|f_i^n(x+h) - f_i^n(x)\| \sim e^{-\lambda_i n}$$

where

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \log \|(D_x f^n)_j\|$$

[Now, the tangent vector $D_x f^n$ evolves in time accordingly with the Jacobean of the map (or flow) f.]

The tangent vector $D_x f^n$ thus measures the distance of two nearby trajectories as a function of time

$$d_n = \|D_x f^n\|$$

For $\Omega = \mathbb{R}^d$, d_n is the Euclidean distance.

Choose an initial error d_0 $(d_0 = h)$

$$\boxed{\frac{d_n}{d_0} = \|D_x f^n\|}$$

Let us define

$$\bar{\lambda}(x, d_0) = \lim_{n\to\infty} \frac{1}{n} \log \frac{d_n}{d_0}$$

Fix a time interval $\tau = m$ and renormalize

$$\bar{d}_n \equiv \frac{d_n}{d_0} \text{ to } 1 \text{ when } n \bmod m = 0$$

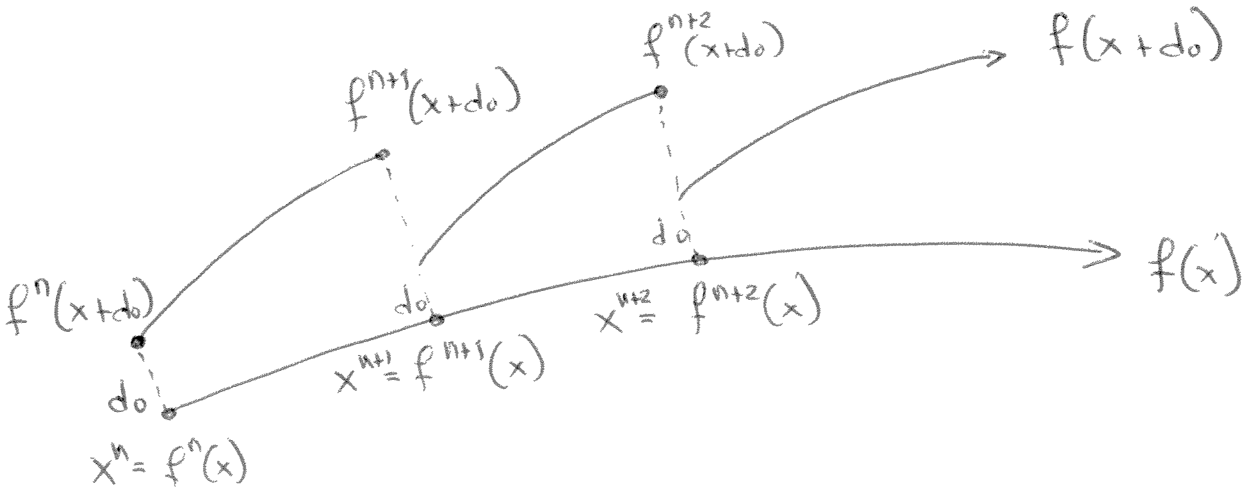
Then

$$\lambda_n = \frac{1}{\tau}\left[\frac{1}{n}\sum_{i=1}^{n} \log \bar{d}_n\right]$$

and

$$\lim_{n\to\infty} \lambda_n = \lambda_0$$

the largest Lyapunov exponent



Logistic map

$$x_{n+1} = \alpha x_n(1-x_n)$$

The derivative of the map is

$$\alpha(1-2x_n)$$

Thus the Lyapunov exponent is

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log|\alpha(1-2x_k)|$$

Thus

$$\lim_{n\to\infty} |f^n(x+h) - f^n(x)| = e^{-\lambda n}$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log |f^{n'}(x)|$$

exists and is called the Lyapunov exponent of f for the measure μ.

Examples (2d)

$$f(x,y) = \begin{cases} (3x, y/4) & , \quad 0 \leq x < 1/3 \\ \left(\frac{3x}{2} - \frac{1}{2}, 1 - y/3\right) & , \quad 1/3 \leq x \leq 1 \end{cases}$$

For this example $D_{f^i(x,y)}f$ are diagonal matrices so they commute

$$D_{(x,y)}f^n h = D_{f^{n-1}(x,y)}f \cdot D_{f^{n-2}(x,y)}f \cdots D_{(x,y)}f h$$

$$= \begin{pmatrix} \prod_{j=0}^{n-1} u(f^j(x,y)) & 0 \\ 0 & \prod_{j=0}^{n-1} v(f^j(x,y)) \end{pmatrix}$$

with

$$u(x,y) = 3\chi_{[0,1/3]}(x) + \frac{3}{2}\chi_{[1/3,1]}(x)$$

$$v(x,y) = \frac{1}{4}\chi_{[0,1/3]}(x) - \frac{1}{3}\chi_{[1/3,1]}(x)$$

applying Birkhoff's ergodic theorem and since u and v do not depend on y,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |u(f^j(x,y))| = \left(\frac{1}{3}\right)\log 3 + \left(\frac{2}{3}\right)\log\frac{3}{2}$$
$$= \log\left(\left(27/4\right)^{1/3}\right)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |v(f^j(x,y))| = \left(\frac{1}{3}\right)\log 4 - \left(\frac{2}{3}\right)\log 3$$
$$= \log\left((36)^{-1/3}\right)$$

Thus

$$D_{(x,y)} f^n \approx \begin{pmatrix} \left(\frac{27}{4}\right)^{n/3} & 0 \\ 0 & (36)^{-n/3} \end{pmatrix}$$

expanding and contracting directions