Lecture 08: Hartman-Grobman theorem and normal forms

Introduction and notation

The expounded material can be found in

- Chapter 2 of [3]
- Chapter 2 of [7]
- Chapter 5 of [1]

As usual we suppose that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{f} \circ \boldsymbol{\phi}_t \tag{0.1}$$

is driven by a vector field **sufficiently smooth** to guarantee the existence of a flow $\boldsymbol{\Phi} \colon \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

1 Topological conjugation

In the discussion of the rectification theorem we introduced the notion of conjugation by diffeomorphism of two dynamical. A weaker notion is

Definition 1.1. *We say that (0.1) is* ***topologically conjugated*** *to*

$$\dot{\boldsymbol{\psi}}_t = g(\boldsymbol{\psi}_t) \tag{1.1}$$

if there exists open sets $\mathcal{U}$ and $\mathcal{W}$ in $\mathbb{R}^n$ and an ***homeomorphism***

$$\boldsymbol{h} \colon \mathcal{U} \to \mathcal{W} \tag{1.2}$$

such that the flows expressing the fundamental solutions of, respectively, (0.1) and (1.1) satisfy

$$\boldsymbol{h} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) = \boldsymbol{\Psi}_t \circ \boldsymbol{h}(\boldsymbol{x}) \tag{1.3}$$

for all $\boldsymbol{x} \in \mathcal{U}$ and $\boldsymbol{h}(\boldsymbol{x}) \in \mathcal{W}$.

An even weaker notion is that of topological equivalence which instead posits the existence of a map

$$f \colon \mathbb{R} \mapsto \mathbb{R} \tag{1.4}$$

such that

$$\boldsymbol{h} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) = \boldsymbol{\Psi}_{f(t)} \circ \boldsymbol{h}(\boldsymbol{x}) \tag{1.5}$$

Equivalence reduces to conjugacy if

$$f(t) = t \tag{1.6}$$

2 Hartman-Grobman theorem

Theorem 2.1. *Let $\boldsymbol{x}_\star$ be a fixed point of (0.1) which we suppose to be driven by a* ***non-linear*** *vector field $\boldsymbol{f} \in \mathrm{C}^r(\mathbb{R}^n;\mathbb{R}^n)$ $r \geq 1$. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ containing $\boldsymbol{x}_\star$, and $\boldsymbol{\Phi}_t$ be the flow of (0.1). Suppose that*

$$\mathsf{A} = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{f})(\boldsymbol{x}_\star) \tag{2.1}$$

has no eigenvalue with zero real part. Then there exists a homeomorphism

$$\boldsymbol{h}\colon \mathcal{U} \mapsto \mathcal{W} \tag{2.2}$$

where $\mathcal{W}$ is an open set V containing $\boldsymbol{x}_\star$ such that for each $\boldsymbol{x} \in \mathcal{U}$, there is an open interval I containing zero such that for all $\boldsymbol{x} \in \mathcal{U}$ and $t \in I$

$$\boldsymbol{h} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) = \boldsymbol{\Psi}_t \circ \boldsymbol{h}(\boldsymbol{x}) \tag{2.3}$$

where $\boldsymbol{\Psi}_t$ is the flow of

$$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t \tag{2.4}$$

Proof. We only present the **idea** of the proof in 3-steps. In order to neaten the notation we assume, as usual, that

$$\boldsymbol{x}_\star = 0 \tag{2.5}$$

and that

$$\mathsf{A} = \mathsf{A}_u \oplus \mathsf{A}_s \tag{2.6}$$

with $\mathsf{A}_u \in \mathrm{End}(\mathrm{R}^{n+})$, $\mathsf{A}_s \in \mathrm{End}(\mathrm{R}^{n-})$. By the hyperbolicity hypothesis

$$n = n_+ + n_- \tag{2.7}$$

(*1*) We write (0.1) as

$$\dot{\boldsymbol{\phi}}_{u;t} = \mathsf{A}_u \cdot \boldsymbol{\phi}_{u;t} + g_u(\boldsymbol{\phi}_{u;t}, \boldsymbol{\phi}_{s;t}) \tag{2.8}$$

$$\dot{\boldsymbol{\phi}}_{s;t} = \mathsf{A}_s \cdot \boldsymbol{\phi}_{s;t} + g_s(\boldsymbol{\phi}_{u;t}, \boldsymbol{\phi}_{s;t}) \tag{2.9}$$

and (2.4) as

$$\dot{\boldsymbol{\psi}}_{u;t} = \mathsf{A}_u \cdot \boldsymbol{\psi}_{u;t} \tag{2.10}$$

$$\dot{\boldsymbol{\psi}}_{s;t} = \mathsf{A}_s \cdot \boldsymbol{\psi}_{s;t} \tag{2.11}$$

From the corresponding flows we define the maps

$$\boldsymbol{F}_u(\boldsymbol{x}) \equiv \boldsymbol{F}_u(\boldsymbol{x}_u, \boldsymbol{x}_s) = \begin{cases} \boldsymbol{\Phi}_{u;t}(\boldsymbol{x}_u, \boldsymbol{x}_s) - e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u & \text{if} \quad \parallel \boldsymbol{x} \parallel \leq r \\ 0 & \text{if} \quad \parallel \boldsymbol{x} \parallel > r \end{cases} \tag{2.12}$$

and similarly

$$\boldsymbol{F}_s(\boldsymbol{x}) \equiv \boldsymbol{F}_s(\boldsymbol{x}_u, \boldsymbol{x}_s) = \begin{cases} \boldsymbol{\Phi}_{s;t}(\boldsymbol{x}_u, \boldsymbol{x}_s) - e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s & \text{if} \quad \parallel \boldsymbol{x} \parallel \leq r \\ 0 & \text{if} \quad \parallel \boldsymbol{x} \parallel > r \end{cases} \tag{2.13}$$

Since for any $\boldsymbol{x}$

$$\partial_{\boldsymbol{x}} \otimes \dot{\boldsymbol{\Phi}}_t(\boldsymbol{x}) = \mathsf{A} \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t)(\boldsymbol{x}) + (\partial_{\boldsymbol{x}} \otimes \boldsymbol{g})(\boldsymbol{\Phi}_t) \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t)(\boldsymbol{x}) \tag{2.14}$$

it follows that for $\boldsymbol{x} = 0$

$$\frac{d}{dt}(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi})_t(\boldsymbol{0}) = \mathsf{A} \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t)(\boldsymbol{0}) \tag{2.15}$$

We have therefore for $i = u, s$

$$\boldsymbol{F}_{i;t}(\boldsymbol{0}) = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{F}_i)(\boldsymbol{0}) \tag{2.16}$$

Moreover for $\parallel \boldsymbol{x} \parallel \leq r$ there is a $K > 0$ such that

$$\parallel (\partial_{\boldsymbol{x}} \otimes \boldsymbol{F}_i)(\boldsymbol{x}) \parallel \leq K \,, \qquad \parallel \boldsymbol{x} \parallel \leq r \quad \& \quad i = u \,, s \tag{2.17}$$

By choosing r sufficiently small, K can be made as small as needed. Then applying the mean value theorem in $\parallel \boldsymbol{x} \parallel \leq r$, there must be a $\boldsymbol{x}_o$ such that $\parallel \boldsymbol{x}_o \parallel \leq \parallel \boldsymbol{x} \parallel \leq r$

$$\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{F}_i(\boldsymbol{0}) + (\partial_{\boldsymbol{x}} \otimes \boldsymbol{F}_i)(\boldsymbol{x}_o) \cdot \boldsymbol{x}_o = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{F}_i)(\boldsymbol{x}_o) \cdot \boldsymbol{x}_o \tag{2.18}$$

whence

$$\parallel \boldsymbol{F}_i(\boldsymbol{x}) \parallel \leq K \parallel \boldsymbol{x} \parallel \leq K \left(\parallel \boldsymbol{x}_u \parallel + \parallel \boldsymbol{x}_s \parallel\right) \qquad i = u, s \tag{2.19}$$

(*2*) Using the above bounds, it is possible to prove that there exists an homeomorphism $\boldsymbol{h}$ mapping a sufficiently small open neighborhood of the origin $\mathcal{U}$ into another $\mathcal{W}$ such that

$$e^{\mathsf{A}} \cdot \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h} \circ \begin{bmatrix} e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u + \boldsymbol{F}_u(\boldsymbol{x}) \\ e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s + \boldsymbol{F}_s(\boldsymbol{x}) \end{bmatrix} \tag{2.20}$$

or equivalently

$$\boldsymbol{h}_u(\boldsymbol{x}_u, \boldsymbol{x}_s) = e^{-\mathsf{A}_u} \cdot \boldsymbol{h}_u(e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u + \boldsymbol{F}_u(\boldsymbol{x}), e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s + \boldsymbol{F}_s(\boldsymbol{x})) \tag{2.21a}$$

$$\boldsymbol{h}_s(\boldsymbol{x}_u, \boldsymbol{x}_s) = e^{-\mathsf{A}_s} \cdot \boldsymbol{h}_s(e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u + \boldsymbol{F}_u(\boldsymbol{x}), e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s + \boldsymbol{F}_s(\boldsymbol{x})) \tag{2.21b}$$

If we set

$$\boldsymbol{y} = e^{\mathsf{A}} \cdot \boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x}) \tag{2.22}$$

we can couch (2.21b) into the form

$$\boldsymbol{h}_s(e^{-\mathsf{A}_u} \cdot \boldsymbol{y}_u + \tilde{\boldsymbol{F}}_u(\boldsymbol{y}), e^{-\mathsf{A}_s} \cdot \boldsymbol{y}_s + \tilde{\boldsymbol{F}}_s(\boldsymbol{y})) = e^{-\mathsf{A}_s} \cdot \boldsymbol{h}_s(\boldsymbol{y}_u, \boldsymbol{y}_s) \tag{2.23}$$

The advantage is that, there exists a $0 < \tilde{K} < 1$ such that

$$\max\{\parallel e^{\mathsf{A}_s} \parallel, \parallel e^{-\mathsf{A}_u} \parallel\} < \tilde{K} \tag{2.24}$$

hence allowing us to prove the existence of the $\boldsymbol{h}_i$ $i = s, u$

$$\boldsymbol{h}_u(\boldsymbol{x}_u, \boldsymbol{x}_s) = e^{-\mathsf{A}_u} \cdot \boldsymbol{h}_u(e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u + \boldsymbol{F}_u(\boldsymbol{x}), e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s + \boldsymbol{F}_s(\boldsymbol{x})) \tag{2.25a}$$

$$\boldsymbol{h}_s(\boldsymbol{y}_u, \boldsymbol{y}_s) = e^{\mathsf{A}_s} \cdot \boldsymbol{h}_s(e^{-\mathsf{A}_u} \cdot \boldsymbol{y}_u + \boldsymbol{F}_u(\boldsymbol{y}), e^{-\mathsf{A}_s} \cdot \boldsymbol{y}_s + \boldsymbol{F}_s(\boldsymbol{y})) \tag{2.25b}$$

by treating both sets of equations in (2.25) on the same footing. Namely we can prove that the sequences

$$\boldsymbol{h}_u^{(n+1)}(\boldsymbol{x}_u, \boldsymbol{x}_s) = e^{-\mathsf{A}_u} \cdot \boldsymbol{h}_u^{(n)}(e^{\mathsf{A}_u} \cdot \boldsymbol{x}_u + \boldsymbol{F}_u(\boldsymbol{x}), e^{\mathsf{A}_s} \cdot \boldsymbol{x}_s + \boldsymbol{F}_s(\boldsymbol{x})) \tag{2.26a}$$

$$\boldsymbol{h}_s^{(n+1)}(\boldsymbol{y}_u, \boldsymbol{y}_s) = e^{\mathsf{A}_s} \cdot \boldsymbol{h}_s(^{(n)}e^{-\mathsf{A}_u} \cdot \boldsymbol{y}_u + \boldsymbol{F}_u(\boldsymbol{y}), e^{-\mathsf{A}_s} \cdot \boldsymbol{y}_s + \boldsymbol{F}_s(\boldsymbol{y})) \tag{2.26b}$$

with initial conditions

$$\boldsymbol{h}_u^{(0)}(\boldsymbol{x}_u, \boldsymbol{x}_s) = \boldsymbol{x}_u \tag{2.27a}$$

$$\boldsymbol{h}_s^{(0)}(\boldsymbol{x}_u, \boldsymbol{x}_s) = \boldsymbol{x}_s \tag{2.27b}$$

are Cauchy sequences with elements in the complete space of continuous functions of $\boldsymbol{x}$. They are therefore convergent. The fixed point of these sequences specify the unique solution of (2.21). See [7] for details.

(3) If an homeomorphism exists at $t = 1$, an homeomorphism exists at any other time. Namely let us define

$$\boldsymbol{H} = \int_0^1 dt_1\, e^{-\mathsf{A}\, t_1} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} \tag{2.28}$$

Then

$$e^{\mathsf{A}\, t} \cdot \boldsymbol{H} = \int_0^1 dt_1\, e^{\mathsf{A}\,(t-t_1)} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} = \int_0^1 dt_1\, e^{\mathsf{A}\,(t-t_1)} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1-t} \circ \boldsymbol{\Phi}_t \tag{2.29}$$

If we show that

$$\int_0^1 dt_1\, e^{\mathsf{A}\,(t-t_1)} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1-t} = \int_0^1 dt_1\, e^{-\mathsf{A}\, t_1} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} \tag{2.30}$$

we have proved the theorem. The proof follows by a direct calculation: the change of integration variable

$$u = t_1 - t \tag{2.31}$$

yields

$$\begin{aligned} \int_0^1 dt_1\, e^{\mathsf{A}\,(t-t_1)} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1-t} &= \int_{-t}^{1-t} du\, e^{-\mathsf{A}\, u} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_u \\ &= \left(\int_{-t}^{0} dt_1 + \int_0^{1-t} dt_1 \right) e^{-\mathsf{A}\, t_1} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} \end{aligned} \tag{2.32}$$

We now observe that

$$\begin{aligned} \int_{-t}^{0} dt_1\, e^{-\mathsf{A}\, t_1} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} &= \int_{-t}^{0} dt_1\, e^{-\mathsf{A}\, t_1} \cdot (e^{\mathsf{A}} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{-1}) \circ \boldsymbol{\Phi}_{t_1} \\ &= \int_{-t}^{0} dt_1\, e^{\mathsf{A}\,(1-t_1)} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1-1} = \int_{1-t}^{1} dt_1\, e^{-\mathsf{A}\, t_1} \cdot \boldsymbol{h} \circ \boldsymbol{\Phi}_{t_1} \end{aligned} \tag{2.33}$$

whence we arrived to

$$e^{\mathsf{A}\, t} \cdot \boldsymbol{H} = \boldsymbol{H} \circ \boldsymbol{\Phi}_t \tag{2.34}$$

□

3 Normal form theory

Normal form theory allows us to gain insight into the meaning and the practical use of the Hartman-Grobman theorem. The idea is to construct explicitly the homeomorphism mapping a non-linear system into one governed by the linearized part of the flow. We won't use here the hyperbolicity assumption for the matrix A associated to the linear flow. Concretely, given

$$\dot{\boldsymbol{\phi}}_{\boldsymbol{t}} = \mathsf{A} \cdot \boldsymbol{\phi}_t + \boldsymbol{g}(\boldsymbol{\phi}_t) \tag{3.1}$$

with $\boldsymbol{g}(\boldsymbol{x}) = O(\| \boldsymbol{x} \|^2)$ and

$$\dot{\boldsymbol{\psi}}_{\boldsymbol{t}} = \mathsf{A} \cdot \boldsymbol{\psi}_t \tag{3.2}$$

respectively defined in neighborhoods $\mathcal{U}$ and $\mathcal{W}$ of the origin we look for

$$\boldsymbol{H} : \mathcal{U} \mapsto \mathcal{W} \tag{3.3}$$

such that for every t for which solutions of (3.1) exist

$$\boldsymbol{\phi}_t = \boldsymbol{H}(\boldsymbol{\psi}_t) \equiv \boldsymbol{\psi}_t + \boldsymbol{h}(\boldsymbol{\psi}_t) \tag{3.4}$$

We have therefore

$$\begin{aligned} &\dot{\boldsymbol{\psi}}_t \cdot \partial_{\boldsymbol{\psi}_t} \boldsymbol{H}(\boldsymbol{\psi}_t) = \\ &\quad (\mathsf{A} \cdot \boldsymbol{\psi}_t) \cdot \partial_{\boldsymbol{\psi}_t} \boldsymbol{H}(\boldsymbol{\psi}_t) = \mathsf{A} \cdot \boldsymbol{H}(\boldsymbol{\psi}_t) + \boldsymbol{g} \circ \boldsymbol{H}(\boldsymbol{\psi}_t) \end{aligned} \tag{3.5}$$

Since the equality must hold independently of t

$$(\mathsf{A} \cdot \boldsymbol{x}) \cdot [\mathsf{I} + \partial_{\boldsymbol{x}} \otimes \boldsymbol{h}(\boldsymbol{x})] = \mathsf{A} \cdot [\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})] + \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})) \tag{3.6}$$

We finally arrive to

$$[1_n \, \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} - \mathsf{A}] \cdot \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})) \tag{3.7}$$

3.1 The homological operator

It is expedient to introduce the differential operator

$$\mathfrak{H}_{\boldsymbol{x}} = 1_n \, \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} - \mathsf{A} \tag{3.8}$$

and refer to it as the "homological" operator. The name is due to its following property

Proposition 3.1. *The homological operator (3.8) maps the space of differentiable homogeneous functions into itself*

Proof. Let $\lambda \in \mathbb{R}_+$ and $\boldsymbol{f}$ a differentiable function such that

$$f(\lambda \, \boldsymbol{x}) = \lambda^m \, f(\boldsymbol{x}) \tag{3.9}$$

The dilation operator

$$\mathfrak{D}_{\boldsymbol{x}} = \boldsymbol{x} \cdot \partial_{\boldsymbol{x}} \tag{3.10}$$

acts on $\boldsymbol{f}$ as

$$\mathfrak{D}_{\boldsymbol{x}} f(\boldsymbol{x}) = m \, f(\boldsymbol{x}) \tag{3.11}$$

We have

$$[\mathfrak{D}_{\boldsymbol{x}} \, , \mathfrak{H}_{\boldsymbol{x}}] := \mathfrak{D}_{\boldsymbol{x}} \mathfrak{H}_{\boldsymbol{x}} - \mathfrak{H}_{\boldsymbol{x}} \mathfrak{D}_{\boldsymbol{x}} = 0 \tag{3.12}$$

which implies the claim. □

3.2 Geometrical interpretation of the homological operator

Geometrically the homological operator is the Lie derivative of the vector field $\boldsymbol{h}$ with respect to $\mathsf{A} \cdot \boldsymbol{x}$.

Definition 3.1. *Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be a pair of smooth vector fields on a manifold $\mathbb{M}_n$ and let $\boldsymbol{\Phi} : \mathbb{R} \times \mathbb{M}_n \mapsto \mathbb{M}_n$ be the local flow generated by the field $\boldsymbol{f}$. The **Lie derivative** of $\boldsymbol{g}$ with respect to $\boldsymbol{f}$ is defined to be the vector field*

$$\mathfrak{L}_{\boldsymbol{x}}[\boldsymbol{f}]\boldsymbol{g}(\boldsymbol{x}) = \lim_{dt \downarrow 0} \frac{\boldsymbol{g} \circ \boldsymbol{\Phi}_{dt}(\boldsymbol{x}) - [(\boldsymbol{\Phi}_{dt})_* \boldsymbol{g}](\boldsymbol{x})}{dt} \tag{3.13}$$

It is straightforward to see that the definition (3.13) implies

$$(\mathfrak{L}_{\boldsymbol{x}}[\boldsymbol{f}]\boldsymbol{g})(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} = [\mathfrak{f}_{\boldsymbol{x}} \, , \mathfrak{g}_{\boldsymbol{x}}] \tag{3.14}$$

where

$$\mathfrak{f}_{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \qquad\qquad \mathfrak{g}_{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \tag{3.15}$$

In order to see this recall that

$$[(\boldsymbol{\Phi}_t)_* \boldsymbol{g}](\boldsymbol{x}) = [(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t) \cdot \boldsymbol{g}] \circ \boldsymbol{\Phi}_{-t}(\boldsymbol{x}) \tag{3.16}$$

We have then

$$\boldsymbol{g} \circ \boldsymbol{\Phi}_{dt}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) + dt \, (\boldsymbol{f} \cdot \partial_{\boldsymbol{x}} \boldsymbol{g})(\boldsymbol{x}) + O(dt^2) \tag{3.17}$$

and

$$[(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t) \cdot \boldsymbol{g}] \circ \boldsymbol{\Phi}_{-t}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) + dt \, [(\partial_{\boldsymbol{x}} \otimes \boldsymbol{f}) \cdot \boldsymbol{g}](\boldsymbol{x}) + O(dt^2) \tag{3.18}$$

The interpretation of the Lie derivative is that of “a Taylor expansion” of $\boldsymbol{g}$ along the orbit of $\boldsymbol{\Phi}$ through $\boldsymbol{x}$. If we now replace in (3.14)

$$\boldsymbol{f}(\boldsymbol{x}) = \mathsf{A} \cdot \boldsymbol{x} \tag{3.19}$$

and

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) \tag{3.20}$$

we find

$$\begin{aligned} \Big[\boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} \, , \boldsymbol{h}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}}\Big] &= \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} - \boldsymbol{h}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \, \boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot \partial_{\boldsymbol{x}} \\ &= [\boldsymbol{x} \cdot \mathsf{A}^{\mathsf{T}} \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{h})(\boldsymbol{x})^{\mathsf{T}} - \mathsf{A} \cdot \boldsymbol{h}(\boldsymbol{x})] \cdot \partial_{\boldsymbol{x}} = (\mathfrak{H}_{\boldsymbol{x}} \boldsymbol{h})(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \end{aligned} \tag{3.21}$$

whence we arrive at

$$\mathfrak{L}_{\boldsymbol{x}}[\mathsf{A} \cdot \boldsymbol{x}]\boldsymbol{h}(\boldsymbol{x}) = \mathfrak{H}_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}) \tag{3.22}$$

3.3 Notation conventions for vector valued homogeneous polynomials

We denote the space of $\mathbb{R}^m$-valued polynomials of n variables

$$\boldsymbol{P}\colon \mathbb{R}^n \mapsto \mathbb{R}^m \tag{3.23}$$

homogeneous of degree k

$$P(\lambda\,\boldsymbol{x}) = \lambda^k\,P(\boldsymbol{x}) \qquad \forall\,\lambda\,\in\,\mathbb{R}_+ \tag{3.24}$$

as $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^m)$. For any $\boldsymbol{x}\in\mathbb{R}^n$, we can write a generic element of $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^m)$ as

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{i=1}^{m} \boldsymbol{e}_i \mathsf{P}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{n\,\text{times}} \tag{3.25}$$

where $\{\boldsymbol{e}_i\}_{i=1}^m$ is the canonical basis of $\mathbb{R}^m$ and $\left\{\mathsf{P}^{(i)}\right\}_{i=1}^m$ is a collection of m coefficients with n indices such that

$$(\mathsf{P}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{n\,\text{times}}) \equiv \left(\sum_{j_1,\dots,j_n=1}^{m} \mathsf{P}^{(i)}_{j_1,\dots,j_n} x^{j_1}x^{j_2}\dots x^{j_n}\right) : \mathbb{R}^n \mapsto \mathbb{R} \qquad \forall\, i = 1,\dots,m \tag{3.26}$$

Example 3.1. Let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.27}$$

and

$$\boldsymbol{P}(\boldsymbol{x}) = \begin{bmatrix} p^{(1)}_{11}x_1^2 + p^{(1)}_{12}x_1x_2 + p^{(1)}_{22}x_2^2 \\ p^{(2)}_{11}x_1^2 + p^{(2)}_{12}x_1x_2 + p^{(2)}_{22}x_2^2 \end{bmatrix} = \sum_{i=1}^{2} \boldsymbol{e}_i \left(p^{(i)}_{11}x_1^2 + p^{(i)}_{12}x_1x_2 + p^{(i)}_{22}x_2^2\right)$$

then

$$\mathsf{P}^{(i)} = \frac{1}{2}\begin{bmatrix} 2\,p^{(i)}_{11} & p^{(i)}_{12} \\ p^{(i)}_{12} & 2\,p^{(i)}_{22} \end{bmatrix} \qquad \Rightarrow \qquad \mathsf{P}^{(i)}_{11} = p^{(i)}_{11} \;\;\&\;\; \mathsf{P}^{(i)}_{12} = \mathsf{P}^{(i)}_{21} = \frac{1}{2}p^{(i)}_{12} \;\;\&\;\; \mathsf{P}^{(i)}_{22} = p^{(i)}_{22} \tag{3.28}$$

More gererally we can think of each of the $\mathsf{P}^{(i)}$'s $i = 1,\dots,m$ as a collection of symmetric tensors of rank n where n is the homogeneity degree of $\boldsymbol{P}$.

3.4 Solution of the homological equation

We can look for the solution of (3.7) by expanding the vector field $\boldsymbol{h}$ in Taylor series. Since the homological operator preserves the homogeneity degree of the functions on which it acts, it does not mix terms of different orders in the Taylor expansion. As a consequence, order by order (3.7) reduces to an algebraic equation for the coefficients of the expansion. Thus if the vector field $\boldsymbol{g}$ admits the expansion

$$\boldsymbol{g}(\boldsymbol{x}) = \sum_{m=2}^{\infty} \boldsymbol{g}_m(\boldsymbol{x}) \tag{3.29}$$

with

$$\boldsymbol{g}_m(\boldsymbol{x}) = \sum_{i=1}^{n} \boldsymbol{e}_i\,\frac{1}{m!}\mathsf{G}^{(i)} : \overbrace{\boldsymbol{x}\otimes\cdots\otimes\boldsymbol{x}}^{m\,\text{times}} \tag{3.30}$$

for $\{\boldsymbol{e}_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$ and $\left\{\mathsf{G}^{(i)}\right\}_{i=1}^m$ the collection of m multi-index coefficients specifying the $O(\| \boldsymbol{x} \|^m)$ term of the Taylor expansion of $\boldsymbol{g}$:

$$\mathsf{G}^{(i)} := (\overbrace{\partial_{\boldsymbol{x}} \otimes \cdots \otimes \partial_{\boldsymbol{x}}}^{m\,\text{times}} \boldsymbol{e}_i \cdot \boldsymbol{g})(\boldsymbol{0}) \tag{3.31}$$

Then we can similarly write

$$\boldsymbol{h}(\boldsymbol{x}) = \sum_{m=2}^{\infty} \boldsymbol{h}_m(\boldsymbol{x}) \tag{3.32}$$

with

$$\boldsymbol{h}_m(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\boldsymbol{e}_i}{m!} \mathsf{H}^{(i)} : \overbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}^{m\,\text{times}} \tag{3.33}$$

We note that

- The evaluation of $\boldsymbol{g}_j$ at $\boldsymbol{x} + \boldsymbol{h}_m$ generates corrections to $\boldsymbol{g}_j(\boldsymbol{x})$ of order of the order $O(\| \boldsymbol{x} \|^{m+j-1})$:

$$\begin{aligned}\boldsymbol{g}_j(\boldsymbol{x} + \boldsymbol{h}_m(\boldsymbol{x})) &= \sum_{i=1}^{n} \frac{\boldsymbol{e}_i}{j!} \mathsf{G}^{(i)} : \overbrace{\boldsymbol{x} + \boldsymbol{h}_m(\boldsymbol{x}) \otimes \cdots \otimes \boldsymbol{x} + \boldsymbol{h}_m(\boldsymbol{x})}^{j\,\text{times}} \\ &= \boldsymbol{g}_j(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\boldsymbol{e}_i}{(j-1)!} \mathsf{G}^{(i)} : \overbrace{\boldsymbol{h}_m(\boldsymbol{x}) \otimes \cdots \otimes \boldsymbol{x}}^{j\,\text{times}} + O(\| \boldsymbol{x} \|^{2\,m+j-2})\end{aligned} \tag{3.34}$$

- For $\| \boldsymbol{x} \| \downarrow 0$ it follows that $O(\| \boldsymbol{x} \|^{m+j-1}) \ll O(\| \boldsymbol{x} \|^m)$ since by hypothesis $j \geq 2$.

Example 3.2. Let us consider the second order case:

$$\boldsymbol{g}_2(\boldsymbol{x}) = \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{e}_i \, \mathsf{G}_2^{(i)} : \boldsymbol{x} \otimes \boldsymbol{x} \tag{3.35}$$

Correspondingly we make the Ansatz

$$\boldsymbol{h}_2(\boldsymbol{x}) = \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{e}_i \, \mathsf{H}_2^{(i)} : \boldsymbol{x} \otimes \boldsymbol{x} \tag{3.36}$$

which yields

$$\begin{aligned}&\mathfrak{H}_{\boldsymbol{x}} \boldsymbol{h}_2(\boldsymbol{x}) \\ &\quad = \sum_{i=1}^{n} \left[\boldsymbol{e}_i \, \boldsymbol{x} \cdot \left(\mathsf{A}^{\mathsf{T}} \frac{\mathsf{H}_2^{(i)} + \mathsf{H}_2^{(i)\mathsf{T}}}{2} \right) \cdot \boldsymbol{x} - \mathsf{A} \cdot \boldsymbol{e}_i \, \frac{1}{2} \mathsf{H}_2^{(i)} : \boldsymbol{x} \otimes \boldsymbol{x} \right] \\ &\quad = \frac{1}{2} \sum_{i=1}^{n} \left[\boldsymbol{e}_i \left(\mathsf{A}^{\mathsf{T}} \mathsf{H}_2^{(i)} + \mathsf{H}_2^{(i)} \mathsf{A} \right) - \mathsf{A} \cdot \boldsymbol{e}_i \, \mathsf{H}_2^{(i)} \right] : \boldsymbol{x} \otimes \boldsymbol{x}\end{aligned} \tag{3.37}$$

We arrive therefore to the equations for the set of symmetric matrices $\left\{\mathsf{H}^{(k)}\right\}_{k=1}^n$:

$$\mathsf{A}^{\mathsf{T}} \mathsf{H}_2^{(k)} + \mathsf{H}_2^{(k)} \mathsf{A} - \sum_{i=1}^{n} (\boldsymbol{e}_k \cdot \mathsf{A} \cdot \boldsymbol{e}_i) \, \mathsf{H}_2^{(i)} = \mathsf{G}_2^k \tag{3.38}$$

In order to simplify the discussion, let us now suppose that A is diagonal in the canonical basis. Equation (3.38) reduces to

$$\mathsf{A}\,\mathsf{H}_2^{(k)} + \mathsf{H}_2^{(k)}\mathsf{A} - a_k\,\mathsf{H}_2^{(k)} = \mathsf{G}_2^k \tag{3.39}$$

In order to solve the matricial equation we can introduce the canonical basis $\{\boldsymbol{e}_{ij}\}_{i,j=1}^{d}$ of $\mathbb{R}^{d\times d}$ whose elements satisfy the relations

$$e_{ij}^{\mathsf{T}} = \boldsymbol{e}_{ji} \qquad \& \qquad \boldsymbol{e}_{ij}\,\boldsymbol{e}_{lk} = \delta_{jl}\boldsymbol{e}_{ik} \tag{3.40}$$

and

$$\mathrm{tr}\boldsymbol{e}_{ij}^{\mathsf{T}}\boldsymbol{e}_{lk} = \delta_{il}\,\mathrm{tr}\boldsymbol{e}_{jk} = \delta_{il}\delta_{jk} \tag{3.41}$$

In the canonical basis we can write a symetric matrix as

$$\mathsf{H}_2^{(k)} = \sum_{i=1}^{d} h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} h_{ij}^{(k)}\frac{\boldsymbol{e}_{ij}+\boldsymbol{e}_{ji}}{2} \tag{3.42}$$

whence

$$\begin{aligned}\mathsf{A}\,\mathsf{H}_2^{(k)} &= \sum_{l=1}^{d} a_l\boldsymbol{e}_{ll}\left(\sum_{i=1}^{d} h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} h_{ij}^{(k)}\frac{\boldsymbol{e}_{ij}+\boldsymbol{e}_{ji}}{2}\right)\\ &= \sum_{i=1}^{d} a_i h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i,l=1}^{d} a_l\,h_{ij}^{(k)}\frac{\delta_{il}\boldsymbol{e}_{lj}+\delta_{jl}\boldsymbol{e}_{li}}{2} = \sum_{i=1}^{d} a_i h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} h_{ij}^{(k)}\frac{a_i\boldsymbol{e}_{ij}+a_j\boldsymbol{e}_{ji}}{2}\end{aligned} \tag{3.43}$$

and similarly

$$\begin{aligned}\mathsf{H}_2^{(k)}\,\mathsf{A} &= \left(\sum_{i=1}^{d} h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} h_{ij}^{(k)}\frac{\boldsymbol{e}_{ij}+\boldsymbol{e}_{ji}}{2}\right)\sum_{l=1}^{d} a_l\boldsymbol{e}_{ll}\\ &= \sum_{i=1}^{d} a_i h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i,l=1}^{d} a_l\,h_{ij}^{(k)}\frac{\delta_{jl}\boldsymbol{e}_{il}+\delta_{il}\boldsymbol{e}_{jl}}{2} = \sum_{i=1}^{d} a_i h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} h_{ij}^{(k)}\frac{a_j\boldsymbol{e}_{ij}+a_i\boldsymbol{e}_{ji}}{2}\end{aligned} \tag{3.44}$$

In view of the above we obtain the equation

$$\sum_{i=1}^{d}(2\,a_i - a_k)h_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d}(a_i+a_j-a_k)\,h_{ij}^{(k)}\frac{\boldsymbol{e}_{ij}+\boldsymbol{e}_{ji}}{2} = \sum_{i=1}^{d} g_i^{(k)}\boldsymbol{e}_{ii} + \sum_{j>i=1}^{d} g_{ij}^{(k)}\frac{\boldsymbol{e}_{ij}+\boldsymbol{e}_{ji}}{2} \tag{3.45}$$

For any fixed k non-resonance conditions are therefore

$$2\,a_i - a_k \neq 0 \qquad \forall\, i = 1,\dots,d \tag{3.46}$$

and

$$a_i + a_j - a_k \neq 0 \qquad \forall\, j > i = 1,\dots,d \tag{3.47}$$

From the complete analysis of the space of vector fields of homogeneity degree two we infer that eigenvectors of the homological operator in the space of vector homogeneous polynomials of degree M take in the basis where A is diagonal the general form

$$\boldsymbol{v}_{\boldsymbol{M},i} = \sum_{i=1}^{d} \boldsymbol{e}_i \prod_{j=1}^{d} x_j^{m_j} \qquad \sum_{j=1}^{d} m_i = M \tag{3.48}$$

where $\boldsymbol{M} = (m_1, \ldots, m_d)$ Namely in such a basis the homological operator takes the form

$$\mathfrak{H} = \sum_{k=1}^{d} (a_k x_k \partial_{x_k} 1 - a_k \boldsymbol{e}_k \otimes \boldsymbol{e}_k) \tag{3.49}$$

so that

$$\mathfrak{H} \boldsymbol{v}_{\boldsymbol{M},i} = \Lambda(\boldsymbol{M}, i)\, \boldsymbol{v}_{\boldsymbol{M},i} \tag{3.50}$$

with

$$\Lambda(\boldsymbol{M}, i) = \sum_{k} (m_k\, a_k - a_i) \tag{3.51}$$

3.5 Fisher-Fock-Bargmann inner product for homogeneous polynomials

In order to analyze systematically the order by order in Taylor series solution of the homological equation it is expedient to introduce a scalar product over $\mathcal{H}_k(\mathbb{R}^n; \mathbb{R}^n)$. The Fisher-Fock-Bargmann inner product is well adapted to our scopes.

Definition 3.2. *Let $P_i \in \mathcal{H}_{m_i}(\mathbb{R}^n; \mathbb{R})$, $i = 1, 2$ for $m_1, m_2 \in \mathbb{N}$:*

$$P_i(\boldsymbol{x}) = \mathsf{P}_i : \overbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}^{m_i \text{ times}} \qquad\qquad \forall\, i = 1, 2 \tag{3.52}$$

The Fisher-Fock-Bargmann inner product over $\mathcal{H}(\mathbb{R}^n; \mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^n; \mathbb{R}^n)$ is defined as

$$\langle P_1\,, P_2 \rangle_{\mathcal{H}(\mathbb{R}^n;\mathbb{R})} = P_1(\partial_{\boldsymbol{x}}) P_2(\boldsymbol{x})|_{\boldsymbol{x}=0} \tag{3.53}$$

Note that $P(\partial_{\boldsymbol{x}})$ means: replace any monomial in $P(\boldsymbol{x})$ with a partial derivative operation of order equal to that of the monomial:

$$P(\boldsymbol{x}) = x_1^3 + x_2 x_3^2 \qquad \Rightarrow \qquad P(\boldsymbol{x}) = \partial_{x_1}^3 + \partial_{x_2} \partial_{x_3}^2 \tag{3.54}$$

An immediate consequence of the definition is that P_1, P_2 are orthogonal if they have different homogeneity degree. Namely

- if $m_1 > m_2$

  $$P_1(\partial_{\boldsymbol{x}}) P_2(\boldsymbol{x}) = 0 \tag{3.55}$$

  because all the monomials in $P_2(\boldsymbol{x})$ are annihilated;

- if $m_1 < m_2$

  $$P_1(\partial_{\boldsymbol{x}}) P_2(\boldsymbol{x})|_{\boldsymbol{x}=0} \in \mathcal{H}_{m_2 - m_1}(\mathbb{R}^n; \mathbb{R}) \tag{3.56}$$

  The definition of the Fisher-Fock-Bargmann inner product requires, however, to evaluate this polynomial in the origin where it equals zero.

In appendix A we show that the definition (3.2) is indeed isomorphic to a scalar product in $\mathbb{L}^2(\mathbb{R}^n)$. Taking for granted that (3.2) is indeed a well defined scalar product, it is straightforward to define a scalar product over $\mathcal{H}(\mathbb{R}^n;\mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$. Namely we observe that a generic element of $\mathcal{H}_m(\mathbb{R}^n;\mathbb{R}^n)$ can be written as

$$\boldsymbol{P}(\boldsymbol{x}) = \sum_{i=1}^{n} \boldsymbol{e}_i \, P^{(i)}(\boldsymbol{x}) \tag{3.57}$$

for $\left\{P^{(i)}(\boldsymbol{x})\right\}_{i=1}^{n}$ a collection of elements of $\mathcal{H}_m(\mathbb{R}^n;\mathbb{R})$. Then we define the inner product

$$\langle \boldsymbol{P}_{m_1} \,, \boldsymbol{P}_{m_2} \rangle_{\mathcal{H}(\mathbb{R}^n;\mathbb{R}^n)} = \sum_{i,j=1}^{n} \boldsymbol{e}_i \cdot \boldsymbol{e}_j \, [P^{(i)}_{m_1:1}(\partial_{\boldsymbol{x}}) \, P^{(j)}_{m_2:2}](\boldsymbol{0}) = \delta_{m_1 \, m_2} \sum_{i=1}^{n} P^{(i)}_{m_2}(\partial_{\boldsymbol{x}}) P^{(i)}_{m_2}(\boldsymbol{x}) \tag{3.58}$$

Again, polynomials of different homogeneity degree are orthogonal. Non-trivial orthogonality conditions emerge therefore only if we fix the homogeneity degree. Once we equipped $\mathcal{H}_m(\mathbb{R}^n;\mathbb{R}^n)$ with the inner product (3.58), we can construct an orthonormal basis $\{\boldsymbol{E}_{m;i}\}_{i=1}^{N(n,m)}$ with

$$N(n,m) = n \, \frac{(n-1+m)}{(n-1)! \, m!} \tag{3.59}$$

of $\mathcal{H}_m(\mathbb{R}^n;\mathbb{R}^n)$ with respect to (3.58). We can then compute the **matrix elements** of the homological operator over $\mathcal{H}_m(\mathbb{R}^n;\mathbb{R}^n)$:

$$\mathsf{H}^{(m)}_{ij} = \langle \boldsymbol{P}_{m:i} \,, \mathfrak{H} \boldsymbol{P}_{m:j} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \tag{3.60}$$

We have therefore [4, 5]

Proposition 3.2. *The Bargmann scalar product (3.58) induces the following decomposition of the space $\mathcal{H}_n$ of $\mathbb{R}^n$-valued homogeneous polynomials of degree n:*

$$\mathcal{H}_m = \mathrm{Im}(\mathfrak{H}\mathcal{H})_m \oplus \mathrm{Ker}(\mathfrak{H}^\dagger \mathcal{H})_m \sim \mathrm{Im}\,\mathsf{H}^{(m)} \oplus \mathrm{Ker}\,\mathsf{H}^{(m)\dagger} \tag{3.61}$$

where

$$\mathrm{Ker}\left\{\mathfrak{H}^\dagger \mathcal{H}_m\left(\mathbb{R}^n\right)\right\} = \left\{\boldsymbol{P}_m \in \mathcal{H}_m \mid e^{-\mathsf{A}^\dagger t} \boldsymbol{P}_m(e^{\mathsf{A}^\dagger t} \cdot \boldsymbol{x}) = \boldsymbol{P}_m(\boldsymbol{x}) \;\; \forall\, t \in \mathbb{R} \;\; \forall \boldsymbol{x} \in \mathbb{R}^n \right\} \tag{3.62}$$

Proof. We proceed in two steps

1. We use the fact that the homological operator is the Lie derivative of the vector field on which it acts **with respect to** the linear vector field $\mathsf{A} \cdot \boldsymbol{x}$. By definition then for any $\boldsymbol{P}_m \in \mathcal{H}_m\left(\mathbb{R}^n\right)$

$$\mathfrak{H}\boldsymbol{P}_m(\boldsymbol{x}) = \left.\frac{d}{dt}\right|_{t=0} e^{-\mathsf{A}\,t} \boldsymbol{P}_m(e^{\mathsf{A}\,t} \cdot \boldsymbol{x}) = -\mathsf{A} \cdot \boldsymbol{P}(\boldsymbol{x}) + \left(\left.\frac{d}{dt} e^{\mathsf{A}\,t} \right|_{t=0} \cdot \boldsymbol{x} \right) \cdot \partial_{\boldsymbol{x}} \boldsymbol{P}(\boldsymbol{x}) \tag{3.63}$$

Thus we can write

$$\begin{aligned} \langle \boldsymbol{P}_{m:a} \,, \mathfrak{H} \boldsymbol{P}_{m:b} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} &= \left.\frac{d}{dt}\right|_{t=0} \langle \boldsymbol{P}_{m:a} \,, e^{-\mathsf{A}\,t} \cdot \boldsymbol{P}_{m:b} \circ e^{\mathsf{A}\,t} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \\ &= \left.\frac{d}{dt}\right|_{t=0} \sum_{i,j=1}^{n} \boldsymbol{e}_i \cdot e^{-\mathsf{A}\,t} \cdot \boldsymbol{e}_j \langle \, P^{(i)}_{m:a} \,, P^{(j)}_{m:b} \circ e^{\mathsf{A}\,t} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \\ &= \left.\frac{d}{dt}\right|_{t=0} \sum_{i,j=1}^{n} \left(e^{-\mathsf{A}^\dagger\,t} \cdot \boldsymbol{e}_i \right) \cdot \boldsymbol{e}_j \langle \, P^{(i)}_{m:a} \,, P^{(j)}_{m:b} \circ e^{\mathsf{A}\,t} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \end{aligned} \tag{3.64}$$

2. We now observe that for any differentiable vector field $\boldsymbol{f}$ and any matrix B the chain of identities

$$\partial_{\boldsymbol{x}} \otimes \boldsymbol{f}(\mathsf{B} \cdot \boldsymbol{x}) = \partial_{\boldsymbol{x}} \otimes (\mathsf{B} \cdot \boldsymbol{x}) \cdot \partial_{\boldsymbol{y}} \otimes \boldsymbol{f}(\boldsymbol{y})|_{\boldsymbol{y}=\mathsf{B}\cdot\boldsymbol{x}} = \mathsf{B}^{\dagger} \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{f})(\mathsf{B} \cdot \boldsymbol{x}) \tag{3.65}$$

Hence we we have

$$\begin{aligned}
&\langle \boldsymbol{P}_{m:a} \,, \mathfrak{H} \boldsymbol{P}_{m:b} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \\
&= \frac{d}{dt}\bigg|_{t=0} \sum_{i,j=1}^{n} \left(e^{-\mathsf{A}^{\dagger}\, t} \cdot \boldsymbol{e}_i \right) \cdot \boldsymbol{e}_j \langle\, P_{m:a}^{(i)} \circ e^{\mathsf{A}^{\dagger}\, t} \,, P_{m:b}^{(j)} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \\
&= \frac{d}{dt}\bigg|_{t=0} \langle\, e^{-\mathsf{A}^{\dagger}\, t} \cdot \boldsymbol{P}_{m:a}^{(i)} \circ e^{\mathsf{A}^{\dagger}\, t} \,, \boldsymbol{P}_{m:b}^{(j)} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \equiv \langle \mathfrak{H}^{\dagger} \boldsymbol{P}_{m:a} \,, \boldsymbol{P}_{m:b} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)}
\end{aligned} \tag{3.66}$$

where now

$$\mathfrak{H}^{\dagger} = \boldsymbol{x} \cdot \mathsf{A} \cdot \partial_{\boldsymbol{x}} - \mathsf{A}^{\dagger} \tag{3.67}$$

3. We have proved that

$$\langle \boldsymbol{P}_{m:a} \,, \mathfrak{H} \boldsymbol{P}_{m:b} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} = \langle \mathfrak{H}^{\dagger} \boldsymbol{P}_{m:a} \,, \boldsymbol{P}_{m:b} \rangle_{\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)} \tag{3.68}$$

hence the left hand side vanishes if the right hand side does. Consider now the problem on $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$:

$$\mathfrak{H} \boldsymbol{h}_k = \boldsymbol{g}_k \tag{3.69}$$

Clearly it admits a unique solution if, given a complete orthonormal basis $\{\boldsymbol{E}_i\}_{i=1}^{N(n,r)}$ of $\mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)$

$$\langle \boldsymbol{E}_i \,, \mathfrak{H}\, \boldsymbol{h}_r \rangle_{\mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)} = \langle \mathfrak{H}^{\dagger} \boldsymbol{E}_i \,, \boldsymbol{h}_k \rangle_{\mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)} = \langle \boldsymbol{E}_i \,, \boldsymbol{g}_r \rangle_{\mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)} \tag{3.70}$$

we have

$$\mathfrak{H}^{\dagger} \boldsymbol{E}_i \neq 0 \qquad\qquad \forall\ i \ \text{ such that } \ \langle \boldsymbol{E}_i \,, \boldsymbol{g}_r \rangle_{\mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)} \neq 0 \tag{3.71}$$

As a consequences we discriminate between elements of $\{\boldsymbol{E}_i\}_{i=1}^{N(n,r)}$ annihilated by $\mathfrak{H}^{\dagger}$ whose span is (3.71) and those which are not and whose span is $\operatorname{Im} \mathfrak{H} \mathcal{H}_r(\mathbb{R}^n;\mathbb{R}^n)$.

□

We are in the position to prove

Proposition 3.3. *For any $r \in \mathbb{N}$ there exists, in a neighborhood of the origin, a polynomial change of variables induced by the formal diffeomorphism*

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(y) = \boldsymbol{y} + \sum_{k=2}^{r} \boldsymbol{h}(y) + O(\parallel \boldsymbol{y} \parallel^{r+1}) \tag{3.72}$$

with $\boldsymbol{h}_k \in \mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$ mapping (3.1) into

$$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t + \sum_{k=2}^{r} \boldsymbol{g}_k^{(\mathrm{res})}(\boldsymbol{\psi}_t) + O(\parallel \boldsymbol{\psi}_t \parallel^{r+1}) \tag{3.73}$$

where

$$\boldsymbol{g}_k^{(\mathrm{res})}(\boldsymbol{x}) \,\in\, \mathrm{Ker} \mathfrak{H} \mathcal{H}_m \qquad \forall\, k \ \mid\ 2 \,\leq\, k \,\leq\, r \tag{3.74}$$

Proof. The proof proceeds by induction.

- Suppose first that we have found a change of variables mapping (3.1) into

$$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t + \sum_{k=2}^{r-1} \boldsymbol{g}_k^{(\mathrm{res})}(\boldsymbol{\psi}_t) + \boldsymbol{g}_r(\boldsymbol{\psi}_t) + O(\| \boldsymbol{\psi}_t \|^{r+1}) \tag{3.75}$$

- Observe that a change of variables of the form

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y}) \tag{3.76}$$

yields

$$\begin{aligned} &\mathsf{A} \cdot [\boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y})] + \sum_{k=2}^{r-1} \boldsymbol{g}_k^{(\mathrm{res})}(\boldsymbol{y} + \boldsymbol{h}_r(\boldsymbol{y})) \\ &\quad = \mathsf{A} \cdot \boldsymbol{y} + \sum_{k=2}^{r-1} \boldsymbol{g}_k^{(\mathrm{res})}(\boldsymbol{y}) + \mathsf{A} \cdot \boldsymbol{h}_r(\boldsymbol{y}) + \sum_{k=2}^{r-1} O(\| \boldsymbol{x} \|^{m+k-1}) \end{aligned} \tag{3.77}$$

In words: resonant terms identified by solving the homological equation up to order $r-1$ in Taylor expansion are not affected by a change of variables of the form (3.76).

- Decompose $\boldsymbol{g}_r$ into a non-resonant and a resonant part:

$$\boldsymbol{g}_r(\boldsymbol{x}) = \boldsymbol{g}_r^{\mathrm{not\ res}}(\boldsymbol{x}) + \boldsymbol{g}_r^{\mathrm{res}}(\boldsymbol{x}) \tag{3.78}$$

In particular, if we can couch the decomposition into the form

$$\boldsymbol{g}_r^{\mathrm{not\ res}}(\boldsymbol{x}) = \sum_{i=1}^{N_1} c_i^{(\mathrm{not\ res})} \boldsymbol{E}_i \tag{3.79a}$$

$$\boldsymbol{g}_r^{\mathrm{res}}(\boldsymbol{x}) = \sum_{i=N_1+1}^{N(n,r)} c_i^{(\mathrm{res})} \boldsymbol{E}_i \tag{3.79b}$$

where

$$\mathfrak{H}^{\dagger} \boldsymbol{E}_i = 0 \qquad\qquad N_1 \leq i \leq N(n,r) \tag{3.80}$$

- The problem

$$\mathfrak{H}\, \boldsymbol{h}_r(\boldsymbol{x}) = \boldsymbol{g}_r^{\mathrm{not\ res}}(\boldsymbol{x}) \tag{3.81}$$

admits a unique solution which we can use to couch (3.1) into the form (3.75) as claimed.

□

3.6 Spectral properties of the homological operator.

The problem of determining the existence of a resonance over $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$ reduces to that of assessing the spectral properties of the homological operator of (3.1) over $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$. To simplify the discussion let us suppose that A has vanishing nilpotent component, then it is straightforward to determine the spectrum of $\mathfrak{H}$

Proposition 3.4. *If* A *in (3.1) is diagonalizable and let* $\{a_i\}_{i=1}^M$ *the collection of distinct eigenvalues in* $\mathrm{Sp}\mathsf{A}$*. Then a resonance occurs over* $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$ *if*

$$a_k = \sum_{i=1}^{M} m_i\, a_i \tag{3.82}$$

for some

$$[m_1, \dots, m_M] \in \mathbb{N}^M \tag{3.83}$$

Proof. By definition $M \leq n$ where the inequality is strict if some eigenvalues have algebraic multiplicity larger than one. Let us suppose this is not the case, the generalization of the proof to the degenerate case being straigthforward. Then $M = n$ so that to each linear independent eigen-vector corresponds a different eigenvalue of A:

$$\mathsf{A} \cdot \boldsymbol{r}_l = a_l\, \boldsymbol{r}_l \qquad l = 1, \dots, n \tag{3.84}$$

Let us suppose to work in the frame of reference where A is diagonal. Let us define for $\sum_{i=1}^n m_i = k$

$$\boldsymbol{E}_{\boldsymbol{I}} = \prod_{i=1}^{n} x_i^{m_i} \boldsymbol{r}_l \in \mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n) \tag{3.85}$$

The $n+1$-tuple

$$\boldsymbol{I} = [m_1, \dots, m_n, l] \tag{3.86}$$

completely specifies the functional form of $\boldsymbol{E}_{\boldsymbol{I}}$. Then, in

$$\mathfrak{H}\boldsymbol{E}_{\boldsymbol{I}} = \left(1 \sum_{i=1}^{n} x_i a_i \partial_{x_i} - \mathsf{A} \right) \boldsymbol{E}_{\boldsymbol{I}} = \left(\sum_{i=1}^{n} m_i\, a_i - a_k \right) \boldsymbol{E}_I \tag{3.87}$$

It follows immediately that $\mathfrak{H}$ has a non-vanishing kernel over $\mathcal{H}_k(\mathbb{R}^n;\mathbb{R}^n)$ if

$$\sum_{i=1}^{n} m_i\, a_i - a_k = 0 \tag{3.88}$$

□

4 Poincaré-Sternberg-Chen theorem and its meaning

Theorem 4.1. *Consider the two equations in* $\mathbb{R}^n$

$$\dot{\boldsymbol{\phi}}_t = \mathsf{A} \cdot \boldsymbol{\phi}_t + \boldsymbol{g}^{(1)}(\boldsymbol{\phi}_t) \tag{4.1a}$$

$$\dot{\boldsymbol{\psi}}_t = \mathsf{A} \cdot \boldsymbol{\psi}_t + \boldsymbol{g}^{(2)}(\boldsymbol{\psi}_t) \tag{4.1b}$$

with A ***hyperbolic**, and* $\boldsymbol{g}^{(i)}(\boldsymbol{x}) = O\left(\| \boldsymbol{x} \|^2\right) \in \mathcal{C}^r(\mathbb{R}^n;\mathbb{R}^n)$*,* $i = 1, 2$ *in a neighborhood of the origin such that*

$$\boldsymbol{g}^{(1)}(\boldsymbol{x}) - \boldsymbol{g}^{(2)}(\boldsymbol{x}) = O(\| \boldsymbol{x} \|^r) \tag{4.2}$$

Then for every $k \geq 2$*, there exists an integer* $N \geq k$ *such that if*

$$r \geq N \tag{4.3}$$

there exists a map $\boldsymbol{h} \in \mathcal{C}^k(\mathbb{R}^n;\mathbb{R}^n)$ *turning (4.1a), (4.1b) into one another.*

In order to interpret the result suppose

$$\boldsymbol{g}^{(2)}(\boldsymbol{x}) = 0 \tag{4.4}$$

Then the theorem states that we can turn a non-linear system into its linearization around an hyperbolic fixed point by means of a k-times differentiable map if we can show that no **resonant term** occurs up to a certain integer N

$$N = N(\mathsf{A}, n, k) \tag{4.5}$$

i.e. depending upon the detailed form of the linearized dynamics, the number of spatial dimensions n and the smoothness we require for the transformation $\boldsymbol{h}$.

A Scalar product and completeness relations

We can gain some intuition about the Fisher-Fock-Bargmann scalar product between homogeneous polynomials by relating it to a standard $\mathbb{L}^2(\mathrm{R}^n)$ scalar product.

A.1 Hermite polynomials

References on multi-dimensional Hermite polynomials are [6, 8]. See also [2]. Let $\boldsymbol{\alpha}_k = [\alpha_1, \ldots, \alpha_n] \in \mathbb{N}^n$ such that

$$\sum_{i=1}^{n} \alpha_i = k \tag{A.1}$$

To any $\boldsymbol{\alpha}$ we can uniquely associate an Hermite polynomial $H^{\boldsymbol{\alpha}_k}(\boldsymbol{x})\colon \mathbb{R}^n \mapsto \mathbb{R}$ as polynomial eigenvectors of the differential operator

$$\mathcal{G}_\sigma := -\boldsymbol{x} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \sigma^2 \Delta_{\boldsymbol{x}} \qquad \sigma \in \mathbb{R}_+ \tag{A.2}$$

i.e.

$$(\mathcal{G}_\sigma H_\sigma^{\boldsymbol{\alpha}_k})(\boldsymbol{x}) = -k \, H_\sigma^{\boldsymbol{\alpha}_1}(\boldsymbol{x}) \tag{A.3}$$

We will refer to the integer k specified by (A.1) as the order of the Hermite polynomial. In particular, we find that Hermite polynomials are specified by Rodrigues formula

$$H_\sigma^{\boldsymbol{\alpha}_k}(\boldsymbol{x}) = \frac{(-\sigma)^k}{\chi_\sigma(\boldsymbol{x})} \prod_{i=1}^{n} \frac{\partial^{\alpha_i}}{\partial x^{\alpha_i}} \chi_\sigma(\boldsymbol{x}) \tag{A.4}$$

where

$$\chi_\sigma(\boldsymbol{x}) = \frac{e^{-\frac{\|\boldsymbol{x}\|^2}{2\,\sigma^2}}}{(2\,\pi\,\sigma^2)^{d/2}} \tag{A.5}$$

Rodrigues formula implies the $\mathbb{L}^2(\mathbb{R}^n)$-orthogonality of Hermite polynomials with respect to the Gaussian measure

$$\langle H_\sigma^{\boldsymbol{\alpha}_{k_1}}\,, H_\sigma^{\boldsymbol{\alpha}_{k_2}}\rangle_{\mathbb{L}^2_{\chi_\sigma}(\mathbb{R}^n)} = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\,\chi_\sigma(\boldsymbol{x})\,H_\sigma^{\boldsymbol{\alpha}_{k_1}}(\boldsymbol{x})\,H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x}) = \prod_{i=1}^n \alpha_{i;1}!\,\delta_{\alpha_{i;1},\alpha_{i;2}} \tag{A.6}$$

Namely we have

$$\langle H_\sigma^{\boldsymbol{\alpha}_{k_1}}\,, H_\sigma^{\boldsymbol{\alpha}_{k_2}}\rangle_{\mathbb{L}^2_{\chi_\sigma}(\mathbb{R}^n)} = \\ (-\sigma)^{k_1}\int_{\mathbb{R}^d}\mathrm{d}^n\boldsymbol{x}\,\prod_{i=1}^n\left[\frac{\partial^{\alpha_{i;1}}}{\partial x^{\alpha_{i;1}}}\chi_\sigma(\boldsymbol{x})\right]H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x}) = \sigma^{k_1}\int_{\mathbb{R}^d}\mathrm{d}^n\boldsymbol{x}\,\chi_\sigma(\boldsymbol{x})\prod_{i=1}^n\frac{\partial^{\alpha_{i;1}}}{\partial x^{\alpha_{i;1}}}H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x}) \tag{A.7}$$

If we integrate by parts we see immediately that

- If $k_1 > k_2$ there are more derivatives than monomials in $H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x})$ thus the integral must vanish.
- If $k_1 < k_2$ since the scalar product is symmetric in the Hermite polynomials, the integral must vanish for the same reason as before exchanging after the roles of $H_\sigma^{\boldsymbol{\alpha}_{k_1}}(\boldsymbol{x})$ and $H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x})$.
- For $k_1 = k_2$ the integral does not vanish only if the derivatives exactly match the **highest order monomial** contained in $H_\sigma^{\boldsymbol{\alpha}_{k_2}}(\boldsymbol{x})$.

A further important consequence of Rodrigues formula (A.4) and the scalar product (A.6) is the **grading relation**

$$\partial_{x^i} H_\sigma^{\boldsymbol{\alpha}_k}(\boldsymbol{x}) = -\frac{\alpha_i}{\sigma} H_\sigma^{\boldsymbol{\alpha}_k - \boldsymbol{e}_i}(\boldsymbol{x}) \tag{A.8}$$

where we used the notation

$$\boldsymbol{\alpha}_k - \boldsymbol{e}_i = [\alpha_1\,, \dots\,, \alpha_i\,, \dots\,, \alpha_n] - [0\,, \,, \dots\,, 1\,, \dots\,, \alpha_n] = [\alpha_1\,, \dots\,, \alpha_i - 1\,, \dots\,, \alpha_n] \tag{A.9}$$

A.2 Relation with homogeneous polynomials

Any polynomial of degree k and in particular Hermite polynomials can be written as a linear combination of homogeneous polynomials up to degree k. On the other hand any polynomial P of degree k can be written as a linear combination of Hermite polynomials. The coefficients of the linear combination can be obtained by computing

$$c_{\boldsymbol{\alpha}_{k_i}} = \langle H_\sigma^{\boldsymbol{\alpha}_{k_i}}\,, P_k\rangle_{\mathbb{L}^2_{\chi_\sigma}(\mathbb{R}^n)} \tag{A.10}$$

for all $\boldsymbol{\alpha}$ with $k_i \leq k$. We can exploit this fact to define an $\mathbb{L}^2_{\chi_\sigma}(\mathbb{R}^n)$ scalar product over homogeneous polynomials. This is done based on the following construction

1. Given two Hermite polynomials of same order k, the value of their scalar product is uniquely determined by the monomials of order k entering their expression.
2. Let $P^{\boldsymbol{\alpha}_k}$ the homogeneous polynomial of degree k

$$P^{\boldsymbol{\alpha}_k} = \prod_{i=1}^n x_i^{\alpha_i} \tag{A.11}$$

   It is always possible to write it as the linear combination

$$P^{\boldsymbol{\alpha}_k} = H^{\boldsymbol{\alpha}_k} + \text{linear combination of Hermite polynomials of order}\, l \leq k \tag{A.12}$$

3. We observe that

$$\int_{\mathbb{R}^d} \mathrm{d}x\, \chi_\sigma(\boldsymbol{x}) = 1 \tag{A.13}$$

From the above two observations we have the the chain of identities

$$\langle P_1 \,, P_2\rangle_{\mathcal{H}(\mathbb{R}^n;\mathbb{R})} := [P_1(\partial_{\boldsymbol{x}})P_2](\mathbf{0}) = \sigma^k \langle H_\sigma^{\boldsymbol{\alpha}_{1;k}} \,, H_\sigma^{\boldsymbol{\alpha}_{2;k}}\rangle_{\mathbb{L}^2_{\chi_\sigma}(\mathbb{R}^n)} \tag{A.14}$$

Since the right most quantity is a well defined scalar product so it must be the leftmost one.

A.3 Application: the adjoint of the homological operator

Let us show that we can derive the expression of the adjoint homological operator using (A.6). To that goal is is sufficient to compute

$$\langle H^{\boldsymbol{\alpha}_{1;k}} \,, \mathfrak{D}_{\mathsf{A}} H^{\boldsymbol{\alpha}_{2;k}}\rangle_{\mathbb{L}^2_\chi(\mathbb{R}^n)} := \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\, \chi(\boldsymbol{x})\, H^{\boldsymbol{\alpha}_{k;1}}(\boldsymbol{x})\, \boldsymbol{x}\cdot \mathsf{A}^\dagger \cdot \partial_{\boldsymbol{x}} H^{\boldsymbol{\alpha}_{k;2}}(\boldsymbol{x}) \tag{A.15}$$

having set

$$\sigma = 1 \tag{A.16}$$

and omitted the under-script. Let us introduce the shorthand notation

$$\mathrm{D}^{\boldsymbol{\alpha}_k} = \prod_{i=1}^n \frac{\partial^{\alpha_{i;1}}}{\partial x^{\alpha_{i;1}}} \tag{A.17}$$

then we have

$$\begin{aligned} &\langle H^{\boldsymbol{\alpha}_{1;k}} \,, \mathfrak{D}_{\mathsf{A}} H^{\boldsymbol{\alpha}_{2;k}}\rangle_{\mathbb{L}^2_\chi(\mathbb{R}^n)} \\ &\quad = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x} \,\left[\chi(\boldsymbol{x})\,\partial_{x^i}\frac{1}{\chi(\boldsymbol{x})}\right]\mathrm{D}^{\boldsymbol{\alpha}_{k;1}}\,[\chi(\boldsymbol{x})]\,\mathsf{A}^\dagger_{ij}\partial_{x^j}\left[\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;2}}\chi(\boldsymbol{x})\right] \\ &\quad = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x} \,\left\{\left[\chi(\boldsymbol{x})\,\partial_{x^i}\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;1}} - \mathrm{D}^{\boldsymbol{\alpha}_{k;1}+\boldsymbol{e}_i}\right]\chi(\boldsymbol{x})\right\}\mathsf{A}^\dagger_{ij}\partial_{x^j}\left[\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;2}}\chi(\boldsymbol{x})\right] \end{aligned} \tag{A.18}$$

It is expedient to analyze separately the contributions to the right hand side.

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\; \chi(\boldsymbol{x})\,\partial_{x^i}\left\{\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;1}}\chi(\boldsymbol{x})\right\}\mathsf{A}^\dagger_{ij}\partial_{x^j}\left[\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;2}}\chi(\boldsymbol{x})\right] \\ &\quad = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\; \chi(\boldsymbol{x})\,\partial_{x^i} H^{\boldsymbol{\alpha}_{k;1}}\mathsf{A}^\dagger_{ij}\frac{1}{\chi(\boldsymbol{x})}\left[x_j\,\mathrm{D}^{\boldsymbol{\alpha}_{k;2}} + \mathrm{D}^{\boldsymbol{\alpha}_{k;2}+\boldsymbol{e}_j}\right]\chi(\boldsymbol{x}) \\ &\quad = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\; \chi(\boldsymbol{x})\,[\partial_{x^i} H^{\boldsymbol{\alpha}_{k;1}}]\mathsf{A}^\dagger_{ij}\,x_j H^{\boldsymbol{\alpha}_{k;2}} + \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\; \chi(\boldsymbol{x})\,[\partial_{x^i} H^{\boldsymbol{\alpha}_{k;1}}]\mathsf{A}^\dagger_{ij} H^{\boldsymbol{\alpha}_{k;2}+\boldsymbol{e}_j} \end{aligned} \tag{A.19}$$

Taking into account the grading property (A.8) we see that the second integral vanishes. The second contribution is

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\; \left\{\mathrm{D}^{\boldsymbol{\alpha}_{k;1}+\boldsymbol{e}_i}\chi(\boldsymbol{x})\right\}\mathsf{A}^\dagger_{ij}\partial_{x^j}\left[\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;2}}\chi(\boldsymbol{x})\right] = \\ &\quad -\int \mathrm{d}^n x\; \left\{\mathrm{D}^{\boldsymbol{\alpha}_{k;1}+\boldsymbol{e}_i+\boldsymbol{e}_j}\chi(\boldsymbol{x})\right\}\mathsf{A}^\dagger_{ij}\left[\frac{1}{\chi(\boldsymbol{x})}\mathrm{D}^{\boldsymbol{\alpha}_{k;2}}\chi(\boldsymbol{x})\right] = -\mathsf{A}^\dagger_{ij}\langle H^{\boldsymbol{\alpha}_{k;1}+\boldsymbol{e}_i+\boldsymbol{e}_j} \,, H^{\boldsymbol{\alpha}_{k;2}}\rangle_{\mathbb{L}^2_\chi(\mathbb{R}^n)} = 0 \end{aligned} \tag{A.20}$$

since is the scalar product of Hermite polynomials of different order. We have therefore proved

$$\langle H^{\boldsymbol{\alpha}_{1;k}} \,, \mathfrak{D}_{\mathsf{A}} H^{\boldsymbol{\alpha}_{2;k}}\rangle_{\mathbb{L}^2_\chi(\mathbb{R}^n)} = \int_{\mathbb{R}^d} \mathrm{d}^n\boldsymbol{x}\, \chi(\boldsymbol{x})\, H^{\boldsymbol{\alpha}_{k;2}}(\boldsymbol{x})\, \boldsymbol{x}\cdot\mathsf{A}\cdot\partial_{\boldsymbol{x}} H^{\boldsymbol{\alpha}_{k;1}}(\boldsymbol{x}) := \langle \mathfrak{D}^\dagger_{\mathsf{A}} H^{\boldsymbol{\alpha}_{1;k}} \,, H^{\boldsymbol{\alpha}_{2;k}}\rangle_{\mathbb{L}^2_\chi(\mathbb{R}^n)} \tag{A.21}$$

Using this identity it is straightforward to recover the expression of the adjoint of the homological operator given in the main text.

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