Lecture 07: Invariant manifold theorem

Introduction and notation

The expounded material can be found in

- Chapter 2 of [1]
- Chapter 3 of [3]

As usual we suppose that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{f} \circ \boldsymbol{\phi}_t \tag{0.1}$$

is driven by a vector field **sufficiently smooth** to guarantee the existence of a flow $\boldsymbol{\Phi} \colon \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

Remark 0.1. Since we are dealing with time autonomous systems by a time translation we can always assign the initial condition at time $t = 0$ and identify t as the time elapsed from the moment when the state of the system is given.

1 Local stable and unstable manifold theorem for hyperbolic fixed points

Theorem 1.1. *Let $\boldsymbol{x}_\star$ be a singular* ***hyperbolic*** *point of $\boldsymbol{f} \in \mathrm{C}^r(\mathbb{R}^n; \mathbb{R}^n)$ for $r \geq 2$, in a neighborhood of $\boldsymbol{x}_\star$. We suppose that*

$$\mathsf{A} = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{f})(\boldsymbol{x}_\star) \in \mathrm{End}(\mathbb{R}^n) \tag{1.1}$$

admits

1. *n_+ eigenvalues with strictly positive real part associated to the invariant subspace E_+;*
2. *n_- eigenvalues with strictly negative real part associated to the invariant subspace E_-*

so that

$$n_+ + n_- = n \tag{1.2}$$

Then, in a neighborhood $\mathcal{U}$ of $\boldsymbol{x}_\star$ there exist

- *a unique local invariant* ***unstable*** *manifold $W^u_{\mathrm{loc.}}(\boldsymbol{x}_\star)$ of class C^r of dimension n_+*
    1. *tangent in $\boldsymbol{x}_\star$ to E_+,*

2. *admitting for a suitable coordinate choice the graph representation*

$$W^u_{\mathrm{loc}}(\boldsymbol{x}_\star) = \left\{ \boldsymbol{x}_u \oplus \boldsymbol{x}_s \in \mathbb{R}^{n_+} \times \mathbb{R}^{n_-} \mid \begin{cases} \boldsymbol{x}_s = \boldsymbol{h}^{(s|u)}(\boldsymbol{x}_u) \\ (\partial_{\boldsymbol{x}_u} \otimes \boldsymbol{h}^{(s|u)})(\boldsymbol{x}_\star) = 0 \end{cases} \& \ \parallel \boldsymbol{x}_u \parallel = \textit{small enough} \right\} \tag{1.3}$$

- *a unique local invariant **stable** manifold* $W^s_{\mathrm{loc.}}(\boldsymbol{x}_\star)$ *with dimension* n_-
  1. *tangent in* $\boldsymbol{x}_\star$ *to* E_+,
  2. *admitting for a suitable coordinate choice the graph representation*

$$W^s_{\mathrm{loc}}(\boldsymbol{x}_\star) = \left\{ \boldsymbol{x}_u \oplus \boldsymbol{x}_s \in \mathbb{R}^{n_+} \times \mathbb{R}^{n_-} \mid \begin{cases} \boldsymbol{x}_u = \boldsymbol{h}^{(u|s)}(\boldsymbol{x}_s) \\ (\partial_{\boldsymbol{x}_s} \otimes \boldsymbol{h}^{(u|s)})(\boldsymbol{x}_\star) = 0 \end{cases} \& \ \parallel \boldsymbol{x}_s \parallel = \textit{small enough} \right\} \tag{1.4}$$

Proof. By hypothesis we can always find a similarity transformation such that

$$\mathsf{T}^{-1}\mathsf{A}\mathsf{T} = \begin{bmatrix} \mathsf{A}_u & 0 \\ 0 & \mathsf{A}_s \end{bmatrix} \tag{1.5}$$

Correspondingly initial data of (0.1) can be written as

$$\boldsymbol{x}_u \oplus \boldsymbol{x}_s = \mathsf{T} \cdot \boldsymbol{x} \tag{1.6}$$

and

$$\mathsf{T}\boldsymbol{\Phi} \circ \boldsymbol{x} = \mathsf{T}\boldsymbol{\Phi} \circ (\mathsf{T}^{-1} \cdot \boldsymbol{x}_u \oplus \boldsymbol{x}_s) = \boldsymbol{\Psi}_t \circ (\boldsymbol{x}_u \oplus \boldsymbol{x}_s) = \boldsymbol{\psi}_{u;t} \oplus \boldsymbol{\psi}_{s;t} \tag{1.7}$$

such that

$$\begin{aligned} \dot{\boldsymbol{\psi}}_{u;t} &= \mathsf{A}_u \cdot \boldsymbol{\psi}_{u;t} + \boldsymbol{g}_u(\boldsymbol{\psi}_{u;t}, \boldsymbol{\psi}_{s;t}) \\ \dot{\boldsymbol{\psi}}_{s;t} &= \mathsf{A}_s \cdot \boldsymbol{\psi}_{s;t} + \boldsymbol{g}_s(\boldsymbol{\psi}_{u;t}, \boldsymbol{\psi}_{s;t}) \end{aligned} \tag{1.8}$$

Note that

$$\boldsymbol{g}_u \oplus \boldsymbol{g}_s := \boldsymbol{g} = \mathsf{T}^{-1} \cdot \boldsymbol{f} \circ \mathsf{T} - \mathsf{T}^{-1} \cdot \mathsf{A}\mathsf{T} = O(\parallel \boldsymbol{x} \parallel^2) \tag{1.9}$$

In particular for any $\varepsilon > 0$ we can find a δ such that

$$\parallel \boldsymbol{g}(\boldsymbol{x}_1) - \boldsymbol{g}(\boldsymbol{x}_2) \parallel \leq \varepsilon \parallel \boldsymbol{x}_1 - \boldsymbol{x}_2 \parallel \tag{1.10}$$

The integral expression of this coupled system of equation is

$$\boldsymbol{\psi}_{u;t} = e^{\mathsf{A}_u t} \cdot \boldsymbol{x}_u + \int_0^t dt_1 \, e^{\mathsf{A}_u (t-t_1)} \cdot \boldsymbol{g}_u(\boldsymbol{\psi}_{u;t_1}, \boldsymbol{\psi}_{s;t_1}) \tag{1.11a}$$

$$\boldsymbol{\psi}_{s;t} = e^{\mathsf{A}_s t} \cdot \boldsymbol{x}_s + \int_0^t dt_1 \, e^{\mathsf{A}_s (t-t_1)} \cdot \boldsymbol{g}_s(\boldsymbol{\psi}_{u;t_1}, \boldsymbol{\psi}_{s;t_1}) \tag{1.11b}$$

where by hypothesis for some $K_1, K_2, K_3 > 0$ and any $\boldsymbol{v} \in \mathbb{R}^{n_-}$

$$\parallel e^{\mathsf{A}_s t} \cdot \boldsymbol{v} \parallel_{\mathbb{R}^{n_-}} \leq K_1 \, e^{-K_2 t} \parallel \boldsymbol{v} \parallel_{\mathbb{R}^{n_-}} \tag{1.12}$$

and any $\boldsymbol{v} \in \mathbb{R}^{n_+}$

$$\| e^{\mathsf{A}_s t} \cdot \boldsymbol{v} \|_{\mathbb{R}^{n_+}} \leq K_1 \, e^{K_3 \, t} \, \| \boldsymbol{v} \|_{\mathbb{R}^{n_+}} \tag{1.13}$$

Let us postulate the existence of a map $\boldsymbol{h}$ relating the initial data of a particular solution of (1.11)

$$\boldsymbol{x}_u = \boldsymbol{h} \circ \boldsymbol{x}_s \tag{1.14}$$

so that only n_- of them are linearly independent around the origin. Then we write

$$\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t} = e^{\mathsf{A}_u \, t} \cdot \left[\boldsymbol{h}(\boldsymbol{x}_s) + \int_0^t dt_1 \, e^{-\mathsf{A}_u \, t_1} \cdot \boldsymbol{g}_u(\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t}, \boldsymbol{\psi}^{\boldsymbol{h}}_{s;t}) \right] \tag{1.15a}$$

$$\boldsymbol{\psi}^{\boldsymbol{h}}_{s;t} = e^{\mathsf{A}_s \, t} \cdot \boldsymbol{x}_s + \int_0^t dt_1 \, e^{\mathsf{A}_s \, (t-t_1)} \cdot \boldsymbol{g}_s(\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t}, \boldsymbol{\psi}^{\boldsymbol{h}}_{s;t}) \tag{1.15b}$$

to emphasize the dependence on the particular choice of initial data. Suppose now that the function $\boldsymbol{h}$ solves the integral equation

$$\boldsymbol{h}(\boldsymbol{x}_s) = - \int_0^\infty dt_1 \, e^{-\mathsf{A}_u \, t_1} \cdot \boldsymbol{g}_u(\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t}, \boldsymbol{\psi}^{\boldsymbol{h}}_{s;t}) \tag{1.16}$$

then (1.15) can be couched into the form

$$\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t} = - \int_t^\infty dt_1 \, e^{\mathsf{A}_u \, (t-t_1)} \cdot \boldsymbol{g}_u(\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t}, \boldsymbol{\psi}^{\boldsymbol{h}}_{s;t}) \tag{1.17a}$$

$$\boldsymbol{\psi}^{\boldsymbol{h}}_{s;t} = e^{\mathsf{A}_s \, t} \cdot \boldsymbol{x}_s + \int_0^t dt_1 \, e^{\mathsf{A}_s \, (t-t_1)} \cdot \boldsymbol{g}_s(\boldsymbol{\psi}^{\boldsymbol{h}}_{u;t}, \boldsymbol{\psi}^{\boldsymbol{h}}_{s;t}) \tag{1.17b}$$

If we introduce the kernels

$$\mathsf{K}_{u;t} = e^{\mathsf{A}_u \, t} \oplus \mathsf{0} \qquad \& \qquad \mathsf{K}_{s;t} = \mathsf{0} \oplus \, e^{\mathsf{A}_s \, t} \tag{1.18}$$

we can couch (1.17) into the compact form

$$\boldsymbol{\Psi}_t(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) = \mathsf{K}_{s;t} \cdot \boldsymbol{x} + \int_0^t dt_1 \, \mathsf{K}_{s;t-t_1} \cdot \boldsymbol{g} \circ \boldsymbol{\Psi}_{t_1}(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) - \int_t^\infty dt_1 \, \mathsf{K}_{s;t-t_1} \cdot \boldsymbol{g} \circ \boldsymbol{\Psi}_{t_1}(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) \tag{1.19}$$

This equation can now be solved by successive approximations. In other words, given the recursion

$$\begin{aligned} \boldsymbol{\Psi}^{(n+1)}_t(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) &= \mathsf{K}_{s;t} \cdot \boldsymbol{x} + \\ & \int_0^t dt_1 \, \mathsf{K}_{s;t-t_1} \cdot \boldsymbol{g} \circ \boldsymbol{\Psi}^{(n)}_{t_1}(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) - \int_t^\infty dt_1 \, \mathsf{K}_{s;t-t_1} \cdot \boldsymbol{g} \circ \boldsymbol{\Psi}^{(n)}_{t_1}(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) \end{aligned} \tag{1.20a}$$

$$\boldsymbol{\Psi}^{(0)}_t(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) = \boldsymbol{0} \tag{1.20b}$$

we can show that it converges to a unique fixed point as n tends to infinity. The existence of such fixed point is equivalent to the existence of a unique solution of (1.19). To prove the claim let us proceed by **induction**. Let us hence suppose that for some positive $\tilde{K}_i > 0, \, i = 1, 2$

$$\| \boldsymbol{\Psi}^{(n)}_t(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) - \boldsymbol{\Psi}^{(n-1)}_t(\boldsymbol{h} \circ \boldsymbol{x}_s, \boldsymbol{x}_s) \| \leq \tilde{K}_1 \frac{e^{-\tilde{K}_2 \, t} \, \| \boldsymbol{x}_s \|_{\mathbb{R}^{n_-}}}{2^{n-1}} \tag{1.21}$$

then

$$\begin{aligned} &\| \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) - \boldsymbol{\Psi}_t^{(n)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \| \\ &\quad\le \varepsilon K_1 \int_0^t dt_1\, e^{-K_2(t-t_1)} \; \| \boldsymbol{\Psi}_{t_1}^{(n)} - \boldsymbol{\Psi}_{t_1}^{(n-1)} \| (\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \\ &\quad -\varepsilon K_1 \int_t^\infty dt_1\, e^{K_3(t-t_1)} \; \| \boldsymbol{\Psi}_{t_1}^{(n)} - \boldsymbol{\Psi}_{t_1}^{(n-1)} \| (\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \end{aligned} \tag{1.22}$$

We obtain

$$\begin{aligned} &\| \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) - \boldsymbol{\Psi}_t^{(n)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \| \\ &\quad\le \varepsilon K_1 \tilde{K}_1 \frac{\| \boldsymbol{x}_s \|_{\mathbb{R}^{n-}}}{2^{n-1}} \left[e^{-K_2 t} \frac{e^{(K_2-\tilde{K}_2)t} - 1}{K_2 - \tilde{K}_2} + e^{K_3 t} \frac{e^{-(K_3+\tilde{K}_2)t}}{K_3 + \tilde{K}_2} \right] \\ &\quad\le \varepsilon K_1 \tilde{K}_1 \frac{e^{-\tilde{K}_2 t} \| \boldsymbol{x}_s \|_{\mathbb{R}^{n-}}}{2^{n-1}} \left[\frac{1 - e^{-(K_2-\tilde{K}_2)t}}{K_2 - \tilde{K}_2} + \frac{1}{K_3 + \tilde{K}_2} \right] \end{aligned} \tag{1.23}$$

For suitable choices of ε and $\tilde{K}_2$ we arrive to

$$\| \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) - \boldsymbol{\Psi}_t^{(n)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \| \le \tilde{K}_1 \frac{e^{-\tilde{K}_2 t} \| \boldsymbol{x}_s \|_{\mathbb{R}^{n-}}}{2^n} \tag{1.24}$$

Since

$$\sum_{n=0}^{\infty} \| \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) - \boldsymbol{\Psi}_t^{(n)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \| \le \sum_{n=0}^{\infty} \tilde{K}_1 \frac{e^{-\tilde{K}_2 t} \| \boldsymbol{x}_s \|_{\mathbb{R}^{n-}}}{2^n} < \infty \tag{1.25}$$

it follows that

$$\lim_{n\uparrow\infty} \| \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) - \boldsymbol{\Psi}_t^{(n)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \| = 0 \tag{1.26}$$

which proves that $\left\{ \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \right\}_{n=0}^{\infty}$ is a Cauchy sequence of continuous functions. The space of continuous functions on a closed interval with supremum norm is complete. This means that every Cauchy sequence is convergent. Hence it is possible to prove that

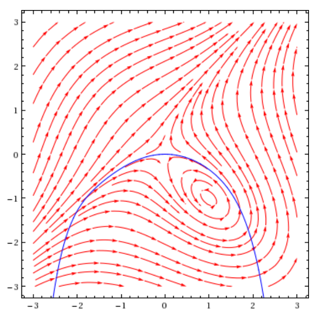
$$\lim_{n\uparrow\infty} \boldsymbol{\Psi}_t^{(n+1)}(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) = \boldsymbol{\Psi}_t(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s) \tag{1.27}$$

uniformly with $\boldsymbol{\Psi}_t(\boldsymbol{h}\circ\boldsymbol{x}_s,\boldsymbol{x}_s)$ differentiable and bounded by a decreasing exponential in t. As a consequence we can also prove that (1.16) is well posed and admits a unique solution specifying the local stable manifold. The existence of the local unstable manifold is obtained similarly by a time reversal operation $t \mapsto -t$ (see e.g. [2] for more details). □

1.1 Example

Let us consider the system

$$\begin{aligned} \dot{\phi}_{1;t} &= -\phi_{1;t} + \phi_{2;t}^2 \\ \dot{\phi}_{2;t} &= \phi_{2;t} + \phi_{1;t}^2 \end{aligned} \tag{1.28}$$



We can determine the local stable manifold of the fixed point in the origin (plotted in blue in the figure above) in two ways.

1.1.1 Iteration method

We define the iteration map

$$\begin{bmatrix} \phi_{1;t}^{(n+1)} \\ \phi_{2;t}^{(n+1)} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ 0 \end{bmatrix} + \int_0^t dt_1 \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} {\phi_{2;t}^{(n)}}^2 \\ {\phi_{1;t}^{(n)}}^2 \end{bmatrix} - \int_t^\infty dt_1 \begin{bmatrix} 0 & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} {\phi_{2;t}^{(n)}}^2 \\ {\phi_{1;t}^{(n)}}^2 \end{bmatrix} \tag{1.29}$$

with

$$\begin{bmatrix} \phi_{1;t}^{(0)} \\ \phi_{2;t}^{(0)} \end{bmatrix} = 0 \tag{1.30}$$

The goal is to determine the unstable manifold using an approximate expression of the flow and the definition (1.16).

1. Order zero

$$\begin{bmatrix} \phi_{1;t}^{(1)} \\ \phi_{2;t}^{(1)} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ 0 \end{bmatrix} \tag{1.31}$$

2. Order one:

$$\begin{bmatrix} \phi_{1;t}^{(1)} \\ \phi_{2;t}^{(1)} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ 0 \end{bmatrix} + \int_0^t dt_1 \begin{bmatrix} e^{-t+t_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_1^2\, e^{-2\,t_1} \end{bmatrix} - \int_t^\infty dt_1 \begin{bmatrix} 0 & 0 \\ 0 & e^{t-t_1} \end{bmatrix} \begin{bmatrix} 0 \\ x_1^2\, e^{-2\,t_1} \end{bmatrix} \tag{1.32}$$

   whence

$$\begin{bmatrix} \phi_{1;t}^{(1)} \\ \phi_{2;t}^{(1)} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ -\frac{x_1^2}{3} e^{-2\,t} \end{bmatrix} \tag{1.33}$$

   We see that (1.16) yields

$$h(x_1) = -\frac{x_1^2}{3} + \dots \tag{1.34}$$

3. Order two:

$$\begin{bmatrix} \phi_{1;t}^{(2)} \\ \phi_{2;t}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ 0 \end{bmatrix} + \int_0^t dt_1 \begin{bmatrix} e^{-t+t_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{x_1^4}{9}\, e^{-4\,t_1} \\ x_1^2\, e^{-2\,t_1} \end{bmatrix} - \int_t^\infty dt_1 \begin{bmatrix} 0 & 0 \\ 0 & e^{t-t_1} \end{bmatrix} \begin{bmatrix} \frac{x_1^4}{9}\, e^{-4\,t_1} \\ x_1^2\, e^{-2\,t_1} \end{bmatrix} \tag{1.35}$$

   yields

$$\begin{bmatrix} \phi_{1;t}^{(2)} \\ \phi_{2;t}^{(2)} \end{bmatrix} = \begin{bmatrix} x_1\, e^{-t} + x_1^4\, \frac{e^{-t} - e^{-4\,t}}{27} \\ -\frac{x_1^2}{3} e^{-2\,t} \end{bmatrix} \tag{1.36}$$

   We see that (1.16) does not receive any new contribution at this order.

4. Order three:

$$
\begin{aligned}
\begin{bmatrix}\phi^{(3)}_{1;t}\\ \phi^{(3)}_{2;t}\end{bmatrix} &= \begin{bmatrix} x_1\, e^{-t} \\ 0\end{bmatrix} + \\
&\int_0^t dt_1 \begin{bmatrix} e^{-t+t_1} & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix} \frac{x_1^4}{9}\, e^{-4\,t_1} \\ x_1^2\, e^{-2\,t} + x_1^8\, \frac{e^{-2\,t}+e^{-8\,t}-2\,e^{-5\,t}}{729} + 2\, x_1^5\, \frac{e^{-2\,t}-e^{-5\,t}}{27}\end{bmatrix} \\
&- \int_t^\infty dt_1 \begin{bmatrix} 0 & 0 \\ 0 & e^{t-t_1}\end{bmatrix} \begin{bmatrix} \frac{x_1^4}{9}\, e^{-4\,t_1} \\ x_1^2\, e^{-2\,t} + x_1^8\, \frac{e^{-2\,t}+e^{-8\,t}-2\,e^{-5\,t}}{729} + 2\, x_1^5\, \frac{e^{-2\,t}-e^{-5\,t}}{27}\end{bmatrix}
\end{aligned} \tag{1.37}
$$

whence

$$
\phi^{(3)}_{1;t} = e^{-t}\, x_1 + x_1^4 \frac{e^{-t} - e^{-4t}}{27} \tag{1.38}
$$

$$
\phi^{(3)}_{2;t} = -\frac{x_1^2}{3}\left(1 + \frac{2\, x_1^3}{27}\right) e^{-2\,t} + \frac{x_1^5}{81} e^{-5\,t} + \frac{x_1^8}{729}\left(\frac{e^{-2\,t}}{3} + \frac{e^{-8\,t}}{9} - \frac{e^{-5\,t}}{3}\right) \tag{1.39}
$$

The expression of the stable manifold at this accuracy is

$$
h(x) = \phi^{(3)}_{2;0} = -\frac{x_1^2}{3}\left(1 + \frac{x_1^3}{27}\right) + O(x_1^8) \tag{1.40}
$$

Note that it is not legitimate to write the explicit expression of the $O(x_1^8)$ monomial.

1.1.2 Graph method

If posit

$$
\phi_{2;t} = h(\phi_{1;t}) \tag{1.41}
$$

we obtain the equation

$$
\dot{\phi}_{1;t} \partial_{\phi_{1;t}} h(\phi_{1;t}) = h(\phi_{1;t}) + \phi^2_{1;t} \tag{1.42}
$$

Since (1.42) must hold independently of time we arrive to

$$
(-x + h^2)\partial_x h = h + x^2 \tag{1.43}
$$

If we postulate

$$
h(x) = c_2\, x^2 + c_3\, x^3 + \dots \tag{1.44}
$$

we obtain

$$
\begin{aligned}
&(-x + c_2^2\, x^4 + 2\, c_2\, c_3\, x^5 + \dots)(2\, c_2\, x + 3\, c_3\, x^2 + 4\, c_4\, x^3 + 5\, c_5 x^4 + \dots) \\
&\quad = (c_2 + 1)\, x^2 + c_3\, x^3 + c_4\, x^4 + c_5\, x^5 + \dots
\end{aligned} \tag{1.45}
$$

whence

$$
c_2 = -\frac{1}{3} \tag{1.46}
$$

and

$$
c_3 = c_4 = 0 \tag{1.47}
$$

Finally we recover

$$
-3\, c_5 + c_2^3 = 0 \qquad \Rightarrow \qquad c_5 = -\frac{1}{81} \tag{1.48}
$$

A Gronwall's Lemma

Lemma A.1. *Let ϕ, α and β continuous and real valued on $[t_1, t_2]$ with*

$$\beta \geq 0 \tag{A.1}$$

and

$$\phi_t \leq \alpha_t + \int_{t_1}^{t} ds\, \beta_t\, \phi_t \tag{A.2}$$

Then

$$\phi_t \leq \alpha_t + \int_{t_1}^{t} ds\, \beta_s\, \alpha_s\, e^{\int_s^t ds_1\, \beta_{s_1}} \tag{A.3}$$

Proof. Let us observe that (A.2) implies for $t \downarrow t_1$

$$\phi_{t_1} \leq \alpha_{t_1} \tag{A.4}$$

- Let us first suppose that ϕ_t and α_t are differentiable. Then we can integrate the inequality

  $$\dot{\phi}_t \leq \dot{\alpha}_t + \beta_t\, \phi_t \tag{A.5}$$

  to obtain

  $$\begin{aligned} \phi_t &\leq e^{\int_{t_1}^t ds\, \beta_s} \phi_{t_1} + \int_{t_1}^{t} ds\, \dot{\alpha}_s e^{\int_s^t ds_1\, \beta_{s_1}} \\ &= e^{\int_{t_1}^t ds\, \beta_s} \phi_{t_1} + \alpha_t - \alpha_{t_1} e^{\int_{t_1}^t ds_1\, \beta_{s_1}} + \int_{t_1}^{t} ds\, \beta_s\, \alpha_s e^{\int_s^t ds_1\, \beta_{s_1}} \end{aligned} \tag{A.6}$$

  whence using (A.4) the claim follows. The general proof proceeds as follows.

- Let us introduce

  $$F_t = \int_{t_1}^{t} ds\, \beta_s\, \phi_s \geq \phi_t - \alpha_t \tag{A.7}$$

  then we must have

  $$\dot{F}_t = \beta_t\, \phi_s \leq \beta_t\, \alpha_t + \beta_t\, F_t \tag{A.8}$$

  Applying the result of the differentiable case we get into

  $$\int_{t_1}^{t} dt_2\, \beta_{t_2}\, \alpha_{t_2}\, e^{\int_{t_2}^t dt_3\, \beta_{t_3}} \geq F_t \geq \phi_t - \alpha_t \tag{A.9}$$

  which is the claim of the proposition.

□

References

[1] N. Berglund. Geometrical theory of dynamical systems, 2007, arXiv:math/0111177.

[2] L. Perko. *Differential Equations and Dynamical Systems*. Springer, 3rd edition, 2006.

[3] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2 of *Texts in applied mathematics*. Springer, 2 edition, 2003.