Lecture 06: invariant manifolds

Introduction and notation

The expounded material can be found in

- Chapter 2 of [1]
- Chapter 3 of [3]
- Chapter 2 of [2]

As usual we suppose that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{f} \circ \boldsymbol{\phi}_t \tag{0.1}$$

is driven by a vector field **sufficiently smooth** to guarantee the existence of a flow $\boldsymbol{\Phi} \colon \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

Remark 0.1. Since we are dealing with time autonomous systems by a time translation we can always assign the initial condition at time $t = 0$ and identify t as the time elapsed from the moment when the state of the system is given.

1 Hyperbolic fixed points

Definition 1.1. *Let $\boldsymbol{x}_\star$ a fixed point of (0.1) in $\mathbb{R}^n$. We say that $\boldsymbol{x}_\star$ is hyperbolic if there exists a similarity transformation* $\mathsf{T} \in \mathrm{End}(\mathbb{R}^n)$ *such that*

$$\mathsf{T}(\partial_{\boldsymbol{x}} \otimes \boldsymbol{f})(\boldsymbol{x}_\star)\mathsf{T}^{-1} = \begin{bmatrix} \mathsf{A}_u & 0 \\ 0 & \mathsf{A}_s \end{bmatrix} \tag{1.1}$$

where $\mathsf{A}_u \in \mathrm{End}(\mathrm{R}^{n+})$, $\mathsf{A}_s \in \mathrm{End}(\mathrm{R}^{n-})$

$$n_+ + n_- = n \tag{1.2}$$

and

1. $\mathrm{Re}\,\mathrm{Sp}\,\mathsf{A}_u > 0$
2. $\mathrm{Re}\,\mathrm{Sp}\,\mathsf{A}_s < 0$

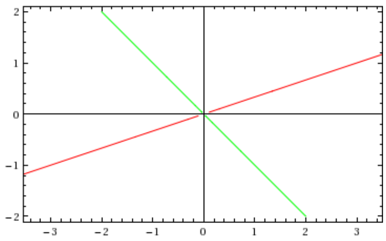
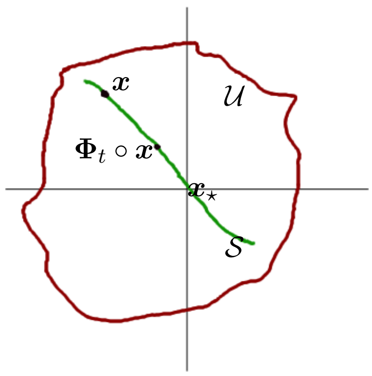


Figure 1.1: Hyperbolic fixed point. In green the linear invariant subspace E_s such that $\mathsf{A}_s\colon \mathrm{E}_s \mapsto \mathrm{E}_s$ over which the flow is contracting. In red the linear invariant subspace E_u such that $\mathsf{A}_u\colon \mathrm{E}_u \mapsto \mathrm{E}_u$ over which the flow is expanding

2 Local manifold theorem

Definition 2.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set. Let $\mathcal{S} \subset \mathbb{R}^n$ have the structure of a differentiable manifold. For $\boldsymbol{x} \in \mathcal{U}$, let $(t_1^x, t_2^x) \ni 0$ be the maximal interval such that $\boldsymbol{\Phi}_t(\boldsymbol{x}) \in \mathcal{U}$ for all $t \in (t_1^x, t_2^x)$. $\mathcal{S}$ is called a* **local invariant manifold** *if*

$$\boldsymbol{\Phi}_t(\boldsymbol{x}) \in \mathcal{S}\ \ \forall\, \boldsymbol{x} \in \mathcal{S} \cap \mathcal{U} \quad \& \quad \forall\, t \in (t_1^x, t_2^x) \tag{2.1}$$



3 Local stable and unstable manifolds for hyperbolic fixed points

Definition 3.1. *Let $\boldsymbol{x}_\star$ be a singular hyperbolic point of $\boldsymbol{f}$, where $\boldsymbol{f}$ is of class C^r, $r \geq 2$, in a neighborhood of $\boldsymbol{x}_\star$. Let $\mathcal{U}$ be aneighbourhood of $\boldsymbol{x}_\star$. We say that*

- *The local* ***stable*** *manifold is the set*

$$W^s_{\mathrm{loc}}(\boldsymbol{x}_\star) = \left\{ \boldsymbol{x} \in \mathcal{U} \mid \lim_{t \uparrow \infty} \boldsymbol{\Phi}_{t,t_o}(\boldsymbol{x}) = \boldsymbol{x}_\star\ \&\ \boldsymbol{\Phi}_{t,t_o}(\boldsymbol{x}) \in \mathcal{U}\ \ \forall\, t \geq 0 \right\} \tag{3.1}$$

- *The local* ***unstable*** *manifold is the set*

$$W^u_{\mathrm{loc}}(\boldsymbol{x}_\star) = \left\{ \boldsymbol{x} \in \mathcal{U} \mid \lim_{t \uparrow -\infty} \boldsymbol{\Phi}_{t,t_o}(\boldsymbol{x}) = \boldsymbol{x}_\star\ \&\ \boldsymbol{\Phi}_{t,t_o}(\boldsymbol{x}) \in \mathcal{U}\ \ \forall\, t \leq 0 \right\} \tag{3.2}$$

Some remarks are in order

- The definition is given for time non-autonomous flows. For autonomous flows $\boldsymbol{\Phi}_{t,t_o} \equiv \boldsymbol{\Phi}_{t-t_o}$ as noticed in the introduction.

- The definitions (3.1), (3.2) are mapped into each other by a time reversal $t \mapsto -t$ operation.
- Requiring the conditions $\boldsymbol{x} \in \mathcal{U}$ and $\boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{x}$ to be simultaneously verified implies that as the time elapses from the moment t_o when the state of the system is assigned, the flow **cannot generate** any "expansion". In other words, with slight abuse of language

$$\boldsymbol{\Phi}_{t,t_o}(\mathcal{U}) \subseteq \mathcal{U} \tag{3.3}$$

- In the linear case

$$\boldsymbol{f}(\boldsymbol{x}) = \mathsf{A} \cdot \boldsymbol{x} \tag{3.4}$$

The local stable and unstable manifolds respectively coincides the linear subspaces $\mathrm{E}_s \subset \mathbb{R}^n$

$$\mathsf{A}_s \colon \mathrm{E}_s \mapsto \mathrm{E}_s \tag{3.5}$$

and

$$\mathsf{A}_u \colon \mathrm{E}_u \mapsto \mathrm{E}_u \tag{3.6}$$

over which the matrix $\mathsf{A} = \mathsf{A}_u \oplus \mathsf{A}_s$ acts as the generator of a contraction as the time respectively grows to $+\infty$ or decreases to $-\infty$.

A general theorem which we will prove in the following lecture guarantees that the terminology "manifolds" is indeed appropriate.

4 Examples

4.1 First example

Let us consider the system

$$\begin{bmatrix} \dot{\phi}_{1;t} \\ \dot{\phi}_{2;t} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \phi_{1;t} \\ \phi_{2;t} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 \\ \phi_{1;t}^n \end{bmatrix} \tag{4.1}$$

for some $n \in \mathbb{N}$ with initial data

$$\begin{bmatrix} \phi_{1;0} \\ \phi_{2;0} \end{bmatrix} = \begin{bmatrix} x_{1;0} \\ x_{2;0} \end{bmatrix} \tag{4.2}$$

Independently of ε (4.9) admits a unique fixed point in the origin of the coordinate system.

4.1.1 Linear case

Let us first set ε to zero. Then the system admits the solution

$$\begin{bmatrix} \phi_{1;t} \\ \phi_{2;t} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ e^{n\,t}\, x_2 \end{bmatrix} \tag{4.3}$$

Hence

$$\mathrm{E}_s = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \& \qquad \mathrm{E}_u = \operatorname{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.4}$$

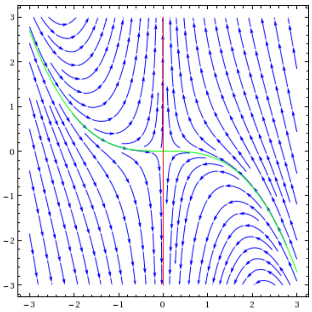


Figure 4.1: Phase-plane plot of the vector field (blue) with stable manifold (green) and unstable manifold (red) for $\varepsilon = 0.5$ and $n = 3$

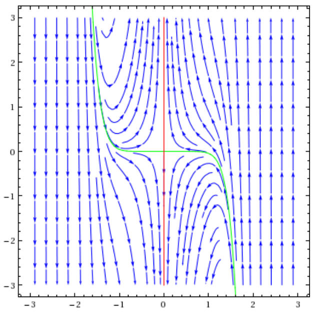


Figure 4.2: Phase-plane plot of the vector field (blue) with stable manifold (green) and unstable manifold (red) for $\varepsilon = 0.5$ and $n = 9$. Note the flattening of the stable manifold in comparison with the case $n = 3$.

4.1.2 Non-linear case

For non-vanishing ε we have instead

$$\begin{bmatrix} \phi_{1;t} \\ \phi_{2;t} \end{bmatrix} = \begin{bmatrix} e^{-t}\, x_1 \\ e^{2\,t}\, x_2 + \varepsilon \int_0^t ds\, e^{2\,(t-s)} e^{-n\,s}\, x_1^n \end{bmatrix} \tag{4.5}$$

After straightforward algebra we see that

$$\phi_{2;t} = e^{2\,t} \left(x_2 + \varepsilon \frac{x_1^n}{n+2} \right) - \varepsilon \frac{x_1^n\, e^{-(n-2)\,t}}{2+n} \tag{4.6}$$

It follows immediately that for $n > 2$

$$W^s_{\text{loc}}(\mathbf{0}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_2 = -\varepsilon \frac{x_1^n}{n+2} \right\} \tag{4.7}$$

whilst

$$W^u_{\text{loc}}(\mathbf{0}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = 0 \right\} \tag{4.8}$$

4.2 Second example

Let us consider now

$$\begin{bmatrix} \dot\phi_{1;t} \\ \dot\phi_{2;t} \\ \dot\phi_{3;t} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{1;t} \\ \phi_{2;t} \\ \phi_{3;t} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 \\ \phi_{1;t}^2 \\ \phi_{1;t}^2 \end{bmatrix} \tag{4.9}$$

with initial data

$$\begin{bmatrix} \phi_{1;0} \\ \phi_{2;0} \\ \phi_{3;0} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.10}$$

Explicit integration yields

$$\begin{bmatrix}\phi_{1;t}\\ \phi_{2;t}\\ \phi_{3;t}\end{bmatrix} = \begin{bmatrix} e^{-t}\,x_1 \\ e^{-t}\,x_2 + x_1^2 \int_0^t ds e^{-t+s} e^{-2\,s} \\ e^{t}\,x_3 + x_1^2 \int_0^t ds e^{t-s} e^{-2\,s} \end{bmatrix} = \begin{bmatrix} e^{-t}\,x_1 \\ e^{-t}\,x_2 + x_1^2 \left(e^{-t} - e^{-2\,t}\right) \\ e^{t}\,x_3 + \frac{x_1^2}{3}\left(e^{t} - e^{-2\,t}\right) \end{bmatrix} \tag{4.11}$$

it follows that

$$W^s_{\mathrm{loc}}(\mathbf{0}) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid x_3 = -\frac{x_1^2}{3} \right\} \tag{4.12}$$

whilst

$$W^u_{\mathrm{loc}}(\mathbf{0}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = x_2 = 0 \right\} \tag{4.13}$$

References

[1] N. Berglund. Geometrical theory of dynamical systems, 2007, arXiv:math/0111177.

[2] L. Perko. *Differential Equations and Dynamical Systems*. Springer, 3rd edition, 2006.

[3] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2 of *Texts in applied mathematics*. Springer, 2 edition, 2003.