Lecture 05: stability of fixed points of non-linear vector field

Introduction and notation

The expounded material can be found in

- Chapter 2 of [1]

As usual we suppose that

$$\dot{\boldsymbol{\phi}}_t = \boldsymbol{f} \circ \boldsymbol{\phi}_t \tag{0.1}$$

is driven by a vector field sufficiently smooth to guarantee the existence of a flow $\boldsymbol{\Phi} \colon \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ ($\mathcal{D}$ stand here as a generic symbol for the state space e.g. $\mathcal{D} = \mathbb{R}^n$) in terms of which we express the solution of (0.1) starting from $\boldsymbol{x}$ at time $t = 0$:

$$\boldsymbol{\phi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{x} \tag{0.2}$$

Definition 0.1. *We say that $\boldsymbol{x}_\star$ is a singular point or $\boldsymbol{f}$ if*

$$\boldsymbol{f}(\boldsymbol{x}_\star) = 0 \tag{0.3}$$

*We will refer to a singular point of a vector field driving an ordinary differential equation such as (0.1) as a **fixed point**.*

1 Conjugation for flows

Proposition 1.1 (Conjugation by a diffeomorphism)**.** *If $\boldsymbol{\Phi}_{t,t_0}$ is the flow of $\boldsymbol{f}$ and $\boldsymbol{\Psi}_{t,t_0}$ is the flow defined by the **push-forward differential***

$$\boldsymbol{u} = \boldsymbol{h}_* \boldsymbol{f} = [(\partial_{\boldsymbol{x}} \otimes \boldsymbol{h}) \cdot \boldsymbol{f}] \circ \boldsymbol{h}^{-1} \tag{1.1}$$

then

$$\boldsymbol{\Psi}_{t,t_0} = \boldsymbol{h} \circ \boldsymbol{\Phi}_{t,t_0} \circ \boldsymbol{h}^{-1} \tag{1.2}$$

Proof. The proof follows from the application of Leibnitz rule

$$\frac{d}{dt}\boldsymbol{\Psi}_{t,t_o} = \frac{d}{dt}\left(\boldsymbol{h} \circ \boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{h}^{-1}\right) = (\dot{\boldsymbol{\Phi}}_{t,t_o} \circ \boldsymbol{h}^{-1}) \cdot (\partial_{\boldsymbol{x}} \boldsymbol{h})(\boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{h}^{-1}) \tag{1.3}$$

Upon noticing that

$$\dot{\boldsymbol{\Phi}}_{t,t_o} \circ \boldsymbol{h}^{-1} = \boldsymbol{f}\left(\boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{h}^{-1}\right) \tag{1.4}$$

we arrive at

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\Psi}_{t,t_o} &= \boldsymbol{f}\left(\boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{h}^{-1}\right) \cdot (\partial_{\boldsymbol{x}} \boldsymbol{h})(\boldsymbol{\Phi}_{t,t_o} \circ \boldsymbol{h}^{-1}) \\ &\boldsymbol{f}\left(\boldsymbol{h}^{-1} \circ \boldsymbol{\Psi}_{t,t_o}\right) \cdot (\partial_{\boldsymbol{x}} \boldsymbol{h})(\boldsymbol{h}^{-1} \circ \boldsymbol{\Psi}_{t,t_o}) = \boldsymbol{h}_* \boldsymbol{f}|_{\boldsymbol{h}^{-1} \circ \boldsymbol{\Psi}_{t,t_o}} = \boldsymbol{u}(\boldsymbol{\Psi}_{t,t_o})\end{aligned} \tag{1.5}$$

and Cauchy-Lipschitz theorem. □

2 Rectification theorem

The rectification theorem explains why we shall be concerned with singular points (or later on, with invariant sets) of the vector field $\boldsymbol{f}$ driving (0.1).

Theorem 2.1. *Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{x}_o) \neq 0$. Then there exists a diffeomorphism $\boldsymbol{h}$ in a neighborhood of $\boldsymbol{x}_o$ such that*

$$\boldsymbol{h}_\star \boldsymbol{f} = [(\partial_{\boldsymbol{x}} \otimes \boldsymbol{h}) \cdot \boldsymbol{f}] \circ \boldsymbol{h}^{-1} \tag{2.1}$$

is constant.

Proof. Modulo a translation and a variable permutation it is non-restrictive to identify $\boldsymbol{x}_o$ with the origin and to assume that the first component of $\boldsymbol{f}$ at the origin is non-vanishing. In formulae:

$$f^1 = \boldsymbol{e}_1 \cdot \boldsymbol{f}(0) \neq 0 \tag{2.2}$$

for $\boldsymbol{e}_1$ the first element of the canonical basis of $\mathbb{R}^n$. Let $\boldsymbol{\Phi}_t$ the flow fundamental solution of (0.1). We can construct the map $\boldsymbol{\rho}$:

$$\boldsymbol{\rho} \colon \mathbb{R} \times \mathbb{R}^{n-1} \mapsto \mathbb{R}^n \tag{2.3}$$

by associating to any point in the neighborhood of the origin of $\mathbb{R}^n$

$$\boldsymbol{x} = [x_1 \, , x_2 \, , \dots \, , x_n]^{\mathsf{t}} \in \mathbb{R}^n \tag{2.4}$$

a vector

$$\boldsymbol{y} = [0 \, , x_2 \, , \dots \, , x_n]^{\mathsf{t}} \in \mathbb{R}^{n-1} \tag{2.5}$$

and then defining $\boldsymbol{\rho}$ as

$$\boldsymbol{\rho}(t, \boldsymbol{y}) := \boldsymbol{\Phi}_t(0, \boldsymbol{y}) \tag{2.6}$$

The map $\boldsymbol{\rho}$ is a diffeomorphism in a neighborhood of the origin of $\mathbb{R} \times \mathbb{R}^{n-1}$. Namely since the flow evaluated at $t = 0$ coincides with the identity map

$$\boldsymbol{\Phi}_0 \circ \boldsymbol{x} = \boldsymbol{x} \tag{2.7}$$

it follows that

- the diffeomorphism coincides with the identity map if $t = 0$

$$\boldsymbol{\rho}(0, \boldsymbol{y}) \equiv \boldsymbol{\Phi}_0(0, \boldsymbol{y}) = [0, x_2, \dots, x_n]^{\mathsf{t}} \tag{2.8}$$

- the differential at the origin is the identity map on $\mathbb{R}^{n-1}$

$$(\partial_{\boldsymbol{y}} \otimes \boldsymbol{\rho})(0, \boldsymbol{0}) = 0 \oplus 1_{n-1} \tag{2.9}$$

As a consequence we have that locally around the origin the diffeomorphism coincides with its differential

$$d\boldsymbol{\rho}(t, \boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{0}) dt + d\boldsymbol{y} \cdot (\partial_{\boldsymbol{y}} \boldsymbol{\phi}_0)(0, \boldsymbol{0}) \tag{2.10}$$

which on its turn is equal to

$$d\boldsymbol{\rho}(t,\boldsymbol{y}) = \boldsymbol{f}(\mathbf{0})dt + \sum_{i=2}^{n} \boldsymbol{e}_i dx^i = \begin{bmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{f}(0) & 0 & 0 & \dots & 0 \\ \boldsymbol{e}_2 \cdot \boldsymbol{f}(0) & 1 & 0 & \dots & 0 \\ \boldsymbol{e}_3 \cdot \boldsymbol{f}(0) & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ \boldsymbol{e}_n \cdot \boldsymbol{f}(0) & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{bmatrix} := \mathsf{R} \cdot \begin{bmatrix} dt \\ d\boldsymbol{y} \end{bmatrix} \tag{2.11}$$

for e_i the i-th element of the canonical basis of $\mathbb{R}^n$. Since $(\boldsymbol{f}(\mathbf{0}), \boldsymbol{e}_2, \dots, \boldsymbol{e}_n)$ forms a basis for $\mathbb{R}^n$, we conclude that the matrix R is non singular and therefore that $d\boldsymbol{\rho}(t,\boldsymbol{y})$ is invertible. By the local inversion theorem $\boldsymbol{\rho}$ is a local diffeomorphism. Let $\boldsymbol{u} = (1, 0, \dots, 0)^{\mathrm{tr}}$ and $\boldsymbol{\Psi}_t$ the flow associated to $\boldsymbol{u}$. Then the following chain of equalities holds

$$\begin{aligned} \boldsymbol{\rho} \circ \boldsymbol{\Psi}_t(\boldsymbol{x}) \equiv \boldsymbol{\rho}(x_1 + t, \boldsymbol{y}) = \\ \boldsymbol{\Phi}_{x_1+t}(0,\boldsymbol{y}) = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_{x_1}(0,\boldsymbol{y}) = \boldsymbol{\Phi}_t \circ \boldsymbol{\rho}(\boldsymbol{x}) \end{aligned} \tag{2.12}$$

where

- the first equality follows from (2.6);
- the second from the definition of flow;
- the third using again (2.6).

We have therefore proved that

$$\boldsymbol{\Phi}_t = \boldsymbol{\rho} \circ \boldsymbol{\Psi}_t \circ \boldsymbol{\rho}^{-1} \tag{2.13}$$

which is the claim of the theorem. □

In other words, whenever $\boldsymbol{f}$ is non-vanishing we can always find a change of coordinates such that the dynamics is locally described by the flow generated by a constant vector field.

3 Stability

The idea of stability: for *any given* distance-value from the fixed point *there exist* initial data in the neighborhood of the fixed point such that after waiting a sufficiently long time their image through the flow is mapped to a distance *equal or smaller* than the pre-assigned one

Definition 3.1. *Let $\boldsymbol{x}_\star$ a fixed point for (0.1). The fixed point is* ***stable*** *if* ***for all*** $\varepsilon > 0$ *we can find a* $\delta(\varepsilon) > 0$ *such that* ***whenever***

$$\parallel \boldsymbol{x} - \boldsymbol{x}_\star \parallel \leq \delta(\varepsilon) \tag{3.1}$$

there exist a $t_* > 0$ *such that for any* $t \geq t_\star$ *we have*

$$\parallel \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \parallel \leq \varepsilon \tag{3.2}$$

A stronger version of stability is

Definition 3.2. *We say that a fixed point $\boldsymbol{x}_\star$ is asymptotically stable if it is stable and there is a δ such that* ***for any*** $\boldsymbol{x}$

$$\| \boldsymbol{x} - \boldsymbol{x}_\star \| \leq \delta \tag{3.3}$$

we have

$$\lim_{t \uparrow \infty} \boldsymbol{\Phi}_t \circ \boldsymbol{x} = \boldsymbol{x}_\star \tag{3.4}$$

We can straightforwardly assess the stability of a fixed point if the vector field $\boldsymbol{f}$ in (0.1) is linear:

$$\boldsymbol{f}(\boldsymbol{x}) = \mathsf{A} \cdot \boldsymbol{x} \tag{3.5}$$

Namely stability is equivalent to the condition that the real part of all the eigenvalues of A are negative definite. In other words the spectrum Sp of A must satisfy

$$\operatorname{Re} \operatorname{Sp} \mathsf{A} \, \leq \, 0 \tag{3.6}$$

In particular:

- the fixed point is stable if

$$\operatorname{Re} \operatorname{Sp} \mathsf{A} \, = \, 0 \tag{3.7}$$

  as in the case of ***elliptic*** fixed points of Hamiltonian flows. Under our hypotheses (0.1) is real-valued. Hence (3.7) is equivalent to say that the eigenvalues of A are purely imaginary. As a consequence the flow describes periodic orbits in the neighborhood of the fixed point, which in the present example coincides with the origin.

- Asymptotic stability in the linear case is equivalent to

$$\operatorname{Re} \operatorname{Sp} \mathsf{A} \, < \, 0 \tag{3.8}$$

  The fixed point in the origin is a sink. It can be approached by the flow with spiraling trajectories if the eigenvalues occur in complex conjugate pairs.

Definition 3.3. *The* ***basin of attraction*** *of an asymptotically stable fixed point $\boldsymbol{x}_\star$ is the set*

$$\left\{ \boldsymbol{x} \, \in \, \mathcal{D} \mid \lim_{t \uparrow \infty} \boldsymbol{\Phi}_t \circ \boldsymbol{x} = \boldsymbol{x}_\star \right\} \tag{3.9}$$

3.1 Instablility

The idea of instability: *there exists* a distance-value from the fixed point such that the image through the flow of *any initial data* in the neighborhood of the fixed point after waiting a sufficiently long time is mapped to a distance *equal or smaller* than the pre-assigned one

Definition 3.4. *The fixed point $\boldsymbol{x}_\star$ is* ***unstable*** *if* ***there exists*** *a $\varepsilon \, > \, 0$ such that* ***for any*** $\delta \, > \, 0$ *there exist an initial data $\boldsymbol{x}$ satisfying*

$$\| \boldsymbol{x} - \boldsymbol{x}_\star \| \leq \, \delta \tag{3.10}$$

and a $t_* \, > \, 0$ such that for any $t \geq t_\star$ we have

$$\| \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \| \geq \, \varepsilon \tag{3.11}$$

4 Lyapunov function

It is convenient to start the discussion by defining

Definition 4.1. *the positive orbit through* $\boldsymbol{x}_o$ *of the flow solution of (0.1) is the set*

$$O_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{\Phi}_t \circ \boldsymbol{x}_o \text{ for some } t \geq 0\} \tag{4.1}$$

The negative orbit is then the set

$$O_- = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{\Phi}_t \circ \boldsymbol{x}_o \text{ for some } t < 0\} \tag{4.2}$$

If the driving vector field $\boldsymbol{f}$ is non-linear asymptotic stability can be straightforwardly proven if $\boldsymbol{f}$ is *gradient*-like. This means that

$$\boldsymbol{f}(\boldsymbol{x}) = -\partial_{\boldsymbol{x}} U \tag{4.3}$$

for some

$$U : \mathbb{R}^n \mapsto \mathbb{R} \tag{4.4}$$

such that

$$(\partial_{\boldsymbol{x}} U)(\boldsymbol{x}_\star) = 0 \tag{4.5a}$$

$$\mathrm{Sp}(\partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} U)(\boldsymbol{x}_\star) < 0 \tag{4.5b}$$

are simultaneously verified. In such a case from

$$\dot{\boldsymbol{\phi}}_t = -\partial_{\boldsymbol{\phi}_t} U \tag{4.6}$$

we can write

$$0 \leq \int_0^T dt \parallel \dot{\boldsymbol{\phi}}_t \parallel^2 = -\int_0^T dt\, \dot{\boldsymbol{\phi}}_t \cdot (\partial_{\boldsymbol{\phi}_t} U)(\boldsymbol{\phi}_t) \tag{4.7}$$

The integrand in the rightmost expression is an exact differential. Hence we must have

$$U \circ \boldsymbol{\Phi}_T(\boldsymbol{x}) \leq U(\boldsymbol{x}) \tag{4.8}$$

where the equality sign occurs only if $\boldsymbol{x} = \boldsymbol{x}_\star$. The fixed point is asymptotically stable. Gradient-like flows are non-generic. We can nevertheless reduce the notion of stability to the existence of a suitable scalar function always decreasing whenever evaluated along the flow:

Theorem 4.1. *Let* $\boldsymbol{x}_\star$ *a singular point of* $\boldsymbol{f}$ *in* $\mathcal{U} \subseteq \mathbb{R}^n$ *and let* $\mathcal{U}_\star = \mathcal{U} / \{\boldsymbol{x}_\star\}$. *If there exists a function*

$$V : \mathcal{U} \mapsto \mathbb{R} \tag{4.9}$$

such that

(*1*) $V(\boldsymbol{x}) > V(\boldsymbol{x}_\star)$ *for all* $\boldsymbol{x} \in \mathcal{U}$;

(*2*) *the derivative of* V *along the flow of* $\boldsymbol{f}$ *is negative definite*

$$\dot{V}(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) = (\boldsymbol{f} \cdot \partial_{\phi_t} V)(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) \leq 0 \quad \forall \boldsymbol{x} \in \mathcal{U}_\star \tag{4.10}$$

then $\boldsymbol{x}_\star$ is stable. Furthermore if

(3) *the derivative of V along the flow of $\boldsymbol{f}$ is **strictly** negative definite on $\mathcal{U}_\star$*

$$\dot{V}(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) < 0 \quad \forall \boldsymbol{x} \in \mathcal{U}_\star \tag{4.11}$$

then $\boldsymbol{x}_\star$ is asymptotically stable.

Proof. The proof consists of two steps.

- Stability: let us consider the **closed** ball

$$\bar{\mathrm{B}}(\boldsymbol{x}_\star, \varepsilon) \in \mathcal{U} \tag{4.12}$$

The boundary of the ball $\mathrm{S} := \bar{\mathrm{B}}(\boldsymbol{x}_\star, \varepsilon)$ is **compact** and therefore the function V must have a minimum on it:

$$V_m = \min_{\boldsymbol{x} \in \mathrm{S}} V(\boldsymbol{x}) \tag{4.13}$$

We can therefore construct an **open** set $\mathcal{V}$ such that

$$\mathcal{V} = \{\boldsymbol{x} \in \mathrm{B}(\boldsymbol{x}_\star, \varepsilon) \mid V(\boldsymbol{x}) < V_m\} \subset \bar{\mathrm{B}}(\boldsymbol{x}_\star, \varepsilon) \tag{4.14}$$

By hypothesis (1) we have $\boldsymbol{x}_\star \in \mathcal{V}$. Hence there must exist a δ such that open ball $\mathrm{B}(\boldsymbol{x}_\star, \delta)$

$$\mathrm{B}(\boldsymbol{x}_\star, \delta) \subset \mathcal{V} \subset \bar{\mathrm{B}}(\boldsymbol{x}_\star, \varepsilon) \tag{4.15}$$

consists of points in the support of V such that

$$V(\boldsymbol{x}) < V_m \quad \forall \boldsymbol{x} \in \mathrm{B}(\boldsymbol{x}_\star, \delta) \tag{4.16}$$

Hypothesis (2) then implies that for all $t \geq 0$

$$V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) < V_m \quad \forall \boldsymbol{x} \in \mathrm{B}(\boldsymbol{x}_\star, \delta) \tag{4.17}$$

which means

$$\boldsymbol{\Phi}_t \circ \boldsymbol{x} \in \mathcal{V} \subset \bar{\mathrm{B}}(\boldsymbol{x}_\star, \varepsilon) \quad \Rightarrow \quad \| \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \| \leq \varepsilon \tag{4.18}$$

Since ε is arbitrary we thus proved that $\boldsymbol{x}_\star$ is stable.

- Asymptotic stability. By (4.18) the positive orbit of any $\boldsymbol{x} \in \mathcal{V}$ is bounded. Hence we can sample the flow along a sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$\boldsymbol{x}_n := \boldsymbol{\Phi}_{t_n}(\boldsymbol{x}) \stackrel{n\uparrow\infty}{\rightarrow} \boldsymbol{y}_\star \tag{4.19}$$

for some

$$\boldsymbol{y}_\star \in \bar{\mathcal{V}} \tag{4.20}$$

Consider now the function

$$\tilde{\mathrm{V}} \colon \mathbb{R} \mapsto \mathbb{R} \tag{4.21}$$

defined by

$$\tilde{V}(t) := V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) \tag{4.22}$$

we notice that

1. it is strictly monotonically decreasing;
2. it admits a convergent sub-sequence

$$\tilde{V}_n = V(\boldsymbol{x}_n) \equiv V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) \stackrel{n\uparrow\infty}{\rightarrow} V(\boldsymbol{y}_\star) \tag{4.23}$$

As a consequence we must have

$$\lim_{t\uparrow\infty} V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) = V(\boldsymbol{y}_\star) \tag{4.24}$$

In order to complete the proof of the theorem, we need only to show that

$$\boldsymbol{y}_\star = \boldsymbol{x}_\star \tag{4.25}$$

necessarily. Indeed let us suppose that (4.25) does not hold true. Then it is possible to construct for some $\delta > 0$ the compact set

$$\mathcal{K} = \left\{\boldsymbol{x} \in \bar{\mathcal{V}} \mid V(\boldsymbol{y}_\star) \leq V(\boldsymbol{x}) \leq V(\boldsymbol{y}_\star) + \delta\right\} \tag{4.26}$$

By condition (3) the inequality

$$V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) - V(\boldsymbol{x}) < 0 \tag{4.27}$$

is always satisfied in $\mathcal{U}_\star$. By (4.24) there must also be a $\tilde{t}$ sufficiently large that

$$\boldsymbol{x} \in \mathcal{K} \quad \& \quad V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) - V(\boldsymbol{x}) < c\,t < 0 \;\; \forall t \geq \tilde{t} \tag{4.28}$$

are simultaneously satisfied. Namely, since $\mathcal{K}$ is compact, we know that there exists the strictly negative constant c

$$c = \max_{\boldsymbol{x} \in \mathcal{K}} \frac{dV}{dt} < 0 \tag{4.29}$$

which provides for (4.28) to be well-posed. But if (4.28) is well-posed then

$$\lim_{t\uparrow\infty} V(\boldsymbol{\Phi}_t \circ \boldsymbol{x}) < \lim_{t\uparrow\infty} \{V(\boldsymbol{x}) + c\,t\} = -\infty \tag{4.30}$$

which contradicts the hypothesis (1). The conclusion is that (4.25) must hold true.

□

Note that (4.25 does not incur in any contradiction because hypothesis (3) allows the derivative of the Lyapunov function to vanish at the singular point.

4.1 Construction of the Lyapunov function for the linearized flow

Proposition 4.1. *Let us suppose that the singular point $\boldsymbol{x}_\star$ of the vector field $\boldsymbol{f}$ is linearly asymptotically stable. Then it is asymptotically stable.*

Proof. Let

$$\boldsymbol{y} := \boldsymbol{x} - \boldsymbol{x}_\star \tag{4.31}$$

denote deviations from the singular point. The linearized flow of (0.1) is

$$\dot{\boldsymbol{\phi}'}_t = \mathsf{A} \cdot \boldsymbol{\phi}'_t \tag{4.32}$$

where

$$\mathsf{A} := (\partial_{\boldsymbol{x}} \otimes \boldsymbol{f})(\boldsymbol{x}_\star) \tag{4.33}$$

Linear asymptotic stability implies the bound

$$\| \mathsf{F}_t \cdot \boldsymbol{y} \| \leq P_n(t)\, e^{-|a|\, t} \, \| \boldsymbol{y} \| \tag{4.34}$$

where by hypothesis

$$a = \max \operatorname{Re} \operatorname{Sp} \mathsf{A} \, < \, 0 \tag{4.35}$$

and $P_n(t)$ is a polynomial in t of degree at most n for n the dimension of the space. To prove the proposition it is sufficient to construct a Lyapunov function V in a neighborhood of $\boldsymbol{x}_\star$. To this goal let us write

$$\boldsymbol{f}(\boldsymbol{x}) = \mathsf{A} \cdot (\boldsymbol{x} - \boldsymbol{x}_\star) + \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{x}_\star) \tag{4.36}$$

where the vector field $\boldsymbol{g}$ is defined by a Taylor expansion of $\boldsymbol{f}$ around the singular point. Hence we can find a constant $K \, \in \, \mathbb{R}_+$ such that

$$\| \, \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{x}_\star) \, \| \leq \, K \, \| \, \boldsymbol{y} \, \|^2 \tag{4.37}$$

We will look for a Lyapunov function of the form

$$V \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) = \langle \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, , \mathsf{Q} \cdot [\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star] \rangle \tag{4.38}$$

for some suitable positive time-independent matrix Q. To prove asymptotic stability we must have

$$\begin{aligned} 0 \, > \, & \dot{V} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) \\ = \, & \langle \dot{\boldsymbol{\Phi}}_t \circ \boldsymbol{x} \, , \mathsf{Q} \cdot [\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star] \rangle + \langle \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, , \mathsf{Q} \cdot \dot{\boldsymbol{\Phi}}_t \circ \boldsymbol{x} \rangle \\ = \, & \langle \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, , (\mathsf{A}^{\mathsf{t}}\mathsf{Q} + \mathsf{Q}\mathsf{A}) \cdot [\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star] \rangle \\ & + 2 \, \langle \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{x}_\star) \, , \mathsf{Q} \cdot [\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star] \rangle + \langle \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{x}_\star) \, , \mathsf{Q} \cdot \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{x}_\star) \rangle \end{aligned} \tag{4.39}$$

using the definition of $\boldsymbol{g}$ we can bound from above $\dot{V}$ as

$$\dot{V} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) \leq \langle \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, , (\mathsf{A}^{\mathsf{t}}\mathsf{Q} + \mathsf{Q}\mathsf{A}) \cdot [\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star] \rangle + 2 \, \tilde{K} \, \| \, \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, \|^3 \tag{4.40}$$

for a suitable choice of a constant $\tilde{K} \, \in \, \mathbb{R}_+$. Since the first term is quadratic in $\boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star$ the claim holds if we can show that under our working hypotheses it is always possible to choose Q such that $\mathsf{A}^{\mathsf{t}}\mathsf{Q} + \mathsf{Q}\mathsf{A}$ is **strictly** negative definite. This is so because linear asymptotic stability implies that

$$\mathsf{Q} = \int_0^\infty dt \, e^{\mathsf{A}^{\mathsf{t}}\, t} e^{\mathsf{A}\, t} \tag{4.41}$$

is well-defined. Furthermore, a direct calculation shows that

$$\mathsf{A}^{\mathsf{t}}\mathsf{Q} + \mathsf{Q}\mathsf{A} = \int_0^\infty dt \, \frac{d}{dt} \, e^{\mathsf{A}^{\mathsf{t}}\, t} e^{\mathsf{A}\, t} = -1_n \tag{4.42}$$

The implication is that if we choose Q as in (4.41), then

$$\dot{V} \circ \boldsymbol{\Phi}_t(\boldsymbol{x}) \leq - \, \| \, \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, \|^2 \, + 2 \, \tilde{K} \, \| \, \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, \|^3 \tag{4.43}$$

is a strict Lyapunov function for any

$$\| \, \boldsymbol{\Phi}_t \circ \boldsymbol{x} - \boldsymbol{x}_\star \, \| < \frac{1}{2 \, K} \tag{4.44}$$

□

References

[1] N. Berglund. Geometrical theory of dynamical systems, 2007, arXiv:math/0111177.