Lecture 04: Hamiltonian linear systems

Introduction

The expounded material can be found in

- Chapter 1 of [3]

1 Hamiltonian dynamical systems

Let us identify $\mathbb{R}^{2\,n}$ as the "phase space" of a dynamical systems described by local coordinates $\boldsymbol{x}$.

Definition 1.1. *We call the vector field*

$$\boldsymbol{f} \colon \mathbb{R} \times \mathbb{R}^{2\,n} \mapsto \mathbb{R}^{2\,n} \tag{1.1}$$

Hamiltonian if

$$\boldsymbol{f}(t, \boldsymbol{x}) = \mathsf{J} \cdot \partial_{\boldsymbol{x}} H(t, \boldsymbol{x}) \tag{1.2}$$

where

$$H \colon \mathbb{R} \times \mathbb{R}^{2\,n} \mapsto \mathbb{R} \tag{1.3}$$

is a differentiable scalar function which we will refer to as the Hamiltonian and

$$\mathsf{J} = -\mathsf{J}^{\mathsf{t}} \tag{1.4}$$

is taken either of the form

$$\mathsf{J} := \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} \tag{1.5}$$

for 1_n the identity map over $\mathbb{R}^n$ or equivalently of the "similar" form

$$\mathsf{J} = \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.6}$$

Note that disregarding of the representation $\mathsf{J}^2 = -1_{2n}$.

Darboux theorem [1, 2] provides a simple physical interpretation of (1.5),(1.6). They amount to choose local coordinates in an flat Euclidean phase space (more generally: "co-tangent bundle" T^*M of a manifold M) such that

1. $\boldsymbol{x} = \boldsymbol{q} \oplus \boldsymbol{p}$ where $\boldsymbol{q}\,, \boldsymbol{p}\, \in \mathbb{R}^n$ are vectors respectively specifying the coordinates of position and momenta of the Hamiltonian system;

2. the same system is a represented as the direct product of n vectors in $\mathbb{R}^2$ so that

$$\boldsymbol{x} = \bigoplus_{i=1}^{n} \begin{bmatrix} q_i \\ p_i \end{bmatrix} \tag{1.7}$$

It is intuitively obvious that there exists in general a similarity transformations connecting these equivalent parametrization of the same physical space.

Example 1.1. A two dimensional physical system obeying to Newton law is described by two positions and two momentum variables. We have

$$\begin{aligned} &\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix} \\ &\qquad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \end{aligned} \tag{1.8}$$

Once we defined an Hamiltonian vector field, it is immediate to define an Hamiltonian ordinary differential equation

$$\dot{\boldsymbol{\chi}}_t = \mathsf{J} \cdot \partial_{\boldsymbol{\chi}_t} H(t, \boldsymbol{\chi}_t) \tag{1.9a}$$

$$\boldsymbol{\chi}_{t_o} = \boldsymbol{x} \tag{1.9b}$$

From the theorem of existence and uniqueness we can immediately say that if H is sufficiently regular, any solution of (1.9) can be described in terms of a *flow*

$$\boldsymbol{X} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2\,n} \mapsto \mathbb{R}^{2\,n} \tag{1.10}$$

such that

$$\boldsymbol{\chi}_t = \boldsymbol{X}(t; t_o, \boldsymbol{x}) \equiv \boldsymbol{X}_{t,t_o}(\boldsymbol{x}) \tag{1.11}$$

1.1 Volume and energy conservation

Hamiltonian systems such as (1.9) are endowed with a special geometric structure, called symplectic, which underlies several important general properties.

Proposition 1.1. *The flow generated by (1.9) is volume preserving.*

Proof. Liouville theorem relates the evolution of volumes to the divergence of the the vector field driving the evolution. For (1.9) we have

$$\partial_{\boldsymbol{x}} \cdot \boldsymbol{f}(t, \boldsymbol{x}) = \partial_{\boldsymbol{x}} \cdot [\mathsf{J} \cdot \partial_{\boldsymbol{x}} H(t, \boldsymbol{x})] = -\mathsf{J} : [(\partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}) H](t, \boldsymbol{x}) = 0 \tag{1.12}$$

since is the trace of a symmetric with an anti-symmetric matrix. □

In (1.12) and throughout these note we use for any $\mathsf{A}\,,\mathsf{B}\,\in\,\mathrm{End}(\mathbb{R}^{2\,n})$ the notation

$$\mathsf{A}:\mathsf{B}:=\mathrm{tr}\mathsf{A}^{\mathsf{t}}\mathsf{B} \tag{1.13}$$

which can be verified to provide a natural notion of scalar product between matrices.

In such a case

Proposition 1.2. *If the function H (1.3) does not depend explicitly upon time it is the preserved by the dynamics.*

Proof. The claim follow by direct calculation

$$\dot{H}\circ\boldsymbol{\chi}_t=\boldsymbol{f}\cdot\partial_{\boldsymbol{\chi}_t}H\circ\boldsymbol{\chi}_t\equiv(\mathsf{J}\cdot\partial_{\boldsymbol{\chi}_t}H)\circ\boldsymbol{\chi}_t\cdot(\partial_{\boldsymbol{\chi}_t}H)\circ\boldsymbol{\chi}_t=\mathsf{J}:[(\partial_{\boldsymbol{\chi}_t}H)\otimes(\partial_{\boldsymbol{\chi}_t}H)]\circ\boldsymbol{\chi}_t=0 \tag{1.14}$$

since is the trace of a symmetric with an anti-symmetric matrix. □

1.2 Symplectomorphisms

An important property of Hamiltonian dynamics is that it is preserved by a large class of diffeomorphisms. Suppose

$$\boldsymbol{\phi}\,\in\,\mathrm{Diff}(\mathbb{R}^{2\,n};\mathbb{R}^{2\,n}) \tag{1.15}$$

such that

$$\boldsymbol{\chi}_t=\boldsymbol{\phi}\circ\boldsymbol{\xi}_t \tag{1.16}$$

then if $\boldsymbol{\chi}_t$ satisfies (1.9) we have

$$\dot{\boldsymbol{\chi}}_t=\dot{\boldsymbol{\xi}}_t\cdot\partial_{\boldsymbol{\xi}_t}\otimes\boldsymbol{\phi}:=\mathsf{F}_t\cdot\dot{\boldsymbol{\xi}}_t \tag{1.17}$$

where $\mathsf{F}_t\in\mathrm{End}(\mathbb{R}^{2\,n})$. On the other hand we must have

$$H(t,\boldsymbol{\chi}_t)=\tilde{H}(t,\boldsymbol{\xi}_t) \tag{1.18}$$

so that

$$(\partial_{\boldsymbol{\chi}_t}H)(t,\boldsymbol{\chi}_t)=(\partial_{\boldsymbol{\chi}_t}\otimes\boldsymbol{\phi}^{-1}\circ\boldsymbol{\chi}_t)^{\mathsf{t}}\cdot(\partial_{\boldsymbol{\xi}_t}\tilde{H})(t,\boldsymbol{\xi}_t)=\mathsf{F}_t^{-1\mathsf{t}}\cdot(\partial_{\boldsymbol{\xi}_t}\tilde{H})(t,\boldsymbol{\xi}_t) \tag{1.19}$$

We have therefore

$$\dot{\boldsymbol{\xi}}_{\boldsymbol{t}}=(\mathsf{F}_t^{-1}\mathsf{J}\mathsf{F}_t^{-1\mathsf{t}})\cdot(\partial_{\boldsymbol{\xi}_t}\tilde{H})(t,\boldsymbol{\xi}_t) \tag{1.20}$$

Definition 1.2. *A matrix $A\in\,\mathbb{R}^{n\times n}$ is said symplectic ($A\in Sp(2n)$) if*

$$A^{\mathsf{t}}\mathsf{J}A=\mathsf{J} \tag{1.21}$$

We have therefore proved that

Proposition 1.3. *Hamiltonian dynamics is invariant in form under the action of symplectomorphisms that under the action of maps $\boldsymbol{\phi}\,\in\,\mathrm{Diff}(\mathbb{R}^{2\,n};\mathrm{R}^{2\,n})$ such that the linearized map*

$$\mathsf{F}=\partial_{\boldsymbol{x}}\otimes\boldsymbol{\phi}\colon\mathbb{R}^{2\,n}\mapsto\mathbb{R}^{2\,n} \tag{1.22}$$

belongs to the symplectic group

$$\mathsf{F}\,\mathsf{J}\,\mathsf{F}^{\mathsf{t}}=\mathsf{J} \tag{1.23}$$

2 Linear Hamiltonian systems

Definition 2.1. *A linear ordinary differential equation over* $\mathbb{R}^{2\,n}$ *is called Hamiltonian if it is amenable to the form*

$$\dot{\boldsymbol{\chi}}_t = \mathsf{J}\,\mathsf{H}_t \cdot \boldsymbol{\chi}_t \tag{2.1a}$$

$$\boldsymbol{\chi}_{t_o} = \boldsymbol{x} \tag{2.1b}$$

where J, H_t *specify endomorphisms of* $\mathbb{R}^n$

$$\mathsf{J}\,,\mathsf{H}_t \colon \mathbb{R} \times \mathbb{R}^{2\,n} \mapsto \mathbb{R}^{2\,n} \tag{2.2}$$

such that

$$\mathsf{H}_t = \mathsf{H}_t^{\mathsf{t}} \tag{2.3}$$

whilst J *is the same as in (1.5), (1.6).*

To neaten the notation we omit the eventual time-dependence of H wherever the choice does not cause ambiguity.

Example 2.1. The harmonic oscillator is a linear Hamiltonian system

$$\begin{aligned} \dot{q}_t &= p_t \\ \dot{p}_t &= -\omega q_t \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} \dot{q}_t \\ \dot{p}_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_t \\ p_t \end{bmatrix} \tag{2.4}$$

Linear Hamiltonian systems are naturally obtained from the linearization of a non-linear Hamiltonian system around one solution. If we denote by $\boldsymbol{\chi'}_t$ the differentiation of solutions of (1.9) with respect to any of the parametric dependence, we obtain

$$\dot{\boldsymbol{\chi}}'_t = \boldsymbol{\chi'}_t \cdot \partial_{\boldsymbol{\chi}_t} \left[\mathsf{J} \cdot \partial_{\boldsymbol{\chi}_t} H \right] := \mathsf{J}\,\mathsf{H}_t \cdot \boldsymbol{\chi'}_t \tag{2.5}$$

with

$$\mathsf{H}_t := (\partial_{\boldsymbol{\chi}_t} \otimes \partial_{\boldsymbol{\chi}_t} H)(\boldsymbol{\chi}_t) \tag{2.6}$$

Remark 2.1. In general, to any symmetric matrix A we can associate a quadratic Hamiltonian

$$\mathsf{A} \cdot \boldsymbol{x} = \frac{1}{2} \partial_{\boldsymbol{x}} [\mathsf{A} : (\boldsymbol{x} \otimes \boldsymbol{x})] = \frac{1}{2} \partial_{\boldsymbol{x}} \langle \boldsymbol{x}\,, \mathsf{A}\boldsymbol{x} \rangle = \partial_{\boldsymbol{x}} H \tag{2.7}$$

For the harmonic oscillator we find

$$H = \frac{p^2}{2} + \omega \frac{q^2}{2} \tag{2.8}$$

2.1 Spectral properties of Symplectic matrices

Proposition 2.1. *The determinant of a symplectic matrix is equal to* ± 1

Proof.

$$1 = \det \mathsf{J} = \det \mathsf{A}^{\mathsf{t}} \mathsf{J} \mathsf{A} = (\det\ \mathsf{A})^2 \tag{2.9}$$

□

Proposition 2.2. *The fundamental solution* F *of (2.1) is a symplectic matrix with determinant equal to one.*

Proof. $\forall\, \boldsymbol{x}_1\, \boldsymbol{x}_2 \in \mathbb{R}^{2\,n}$ one has

$$\begin{aligned}\frac{d}{dt}(\mathsf{J}\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_1)\cdot(\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_2) &\equiv \frac{d}{dt}\langle \mathsf{J}\mathsf{F}_{t,t_o}\boldsymbol{x}_1\,,\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_2\rangle \\ &= -\langle \mathsf{H}_t\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_1\,,\mathsf{F}_{t,t_o}\boldsymbol{x}_2\rangle + \langle \mathsf{J}\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_1\,,\mathsf{J}\mathsf{H}_t\cdot\boldsymbol{x}_2\rangle \\ &= -\langle \mathsf{H}_t\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_1\,,\mathsf{F}_{t,t_o}\boldsymbol{x}_2\rangle + \langle \mathsf{J}^{\mathsf{t}}\mathsf{J}\mathsf{F}_{t,t_o}\cdot\boldsymbol{x}_1\,,\mathsf{H}_t\cdot\boldsymbol{x}_2\rangle = 0\end{aligned} \tag{2.10}$$

We have thus proved that $\mathsf{F}_{t,t_o}\,\mathsf{J}\,\mathsf{F}_{t,t_o}$ is a conserved quantity of the dynamics. Since furthermore

$$\mathsf{F}_{t_o,t_o} = 1_{2\,n} \tag{2.11}$$

we finally have

$$\mathsf{F}_{t,t_o}\,\mathsf{J}\,\mathsf{F}_{t,t_o} = \mathsf{J} \tag{2.12}$$

Finally, $\det 1_{2n} = 1$ implies by continuity of the flow $\det \mathsf{F}_{t,t_o} = 1$ □

Proposition 2.3. *The eigenvalues of a real symplectic matrix occur*

1. *in pairs* $(\lambda, \lambda^\star)$ *if* λ *is on the unit circle* $\mathbb{S}^1$*;*
2. *in pairs* $(\lambda\,,\lambda^{-1})$ *if* $\lambda \in \mathbb{R}$*;*
3. *in quartets* $(\lambda\,,\lambda^\star\,,\lambda^{-1}\,,\lambda^{-1\,\star})$ *if* $\lambda \in \mathbb{C}/(\mathbb{S}^1 \cup \mathbb{R})$

Proof.

$$P(\mathsf{F}) = \det(\mathsf{F} - \lambda 1_{2n}) = \det(\mathsf{J}\,\mathsf{F}\,\mathsf{J}^{\mathsf{t}} - 1_{2n}) = \det(\mathsf{F}^{\mathsf{t}^{-1}} - 1_{2n}) = \det(\mathsf{F}^{-1} - 1_{2n}) \tag{2.13}$$

hence if λ is an eigenvalues so is λ^{-1}. Furthermore, since F is real-valued if $\lambda \in \mathbb{C}$ is an eigenvalue so is λ^*. □

3 Linear periodic Hamiltonian systems

Proposition 3.1. *Let* $\mathsf{H}_t = \mathsf{H}_{t+T}$ *for some* T *and for all* t*. Then the fundamental solution of (2.1) admits a Floquet decomposition in the form*

$$\mathsf{F}_{t,t_o} = \mathsf{P}_{t,t_o}\, e^{\mathsf{J}\,\mathsf{K}\,(t-t_o)} \tag{3.1}$$

where P *is periodic and symplectic,* K *is symmetric and* $e^{\mathsf{J}\,\mathsf{K}\,t}$ *is symplectic.*

Proof. From the general case

$$\mathsf{F}_{t,t_o} = \mathsf{P}_{t,t_o}\, e^{\mathsf{L}\,(t-t_o)} \tag{3.2}$$

Then

$$\mathsf{J} = \mathsf{F}^{\mathsf{t}}_{t,t_o}\mathsf{J}\,\mathsf{F}_{t,t_o} = e^{\mathsf{L}^{\mathsf{t}}\,(t-t_o)}\,\mathsf{P}^{\mathsf{t}}_{t,t_o}\mathsf{J}\mathsf{P}_{t,t_o}\,e^{\mathsf{L}\,(t-t_o)} \qquad \Rightarrow \qquad \mathsf{P}^{\mathsf{t}}_{t,t_o}\mathsf{J}\mathsf{P}_{t,t_o} = e^{-\mathsf{L}^{\mathsf{t}}\,(t-t_o)}\,\mathsf{J}\,e^{-\mathsf{L}\,(t-t_o)} \tag{3.3}$$

Setting $t = t_o + T$ it follows

$$e^{-\mathsf{L}^{\mathsf{t}}\,T}\mathsf{J}e^{-\mathsf{L}\,T} = \mathsf{J} \tag{3.4}$$

Taking the time derivative

$$e^{-\mathsf{L}^{\mathsf{t}}(t-t_o)}(-\mathsf{L}^{\mathsf{t}}\mathsf{J}+\mathsf{J}\mathsf{L})\,e^{-\mathsf{L}\,(t-t_o)} = \dot{\mathsf{P}}^{\mathsf{t}}_{t,t_o}\mathsf{J}\mathsf{P}_{t,t_o} + \mathsf{P}^{\mathsf{t}}_{t,t_o}\mathsf{J}\dot{\mathsf{P}}_{t,t_o} \tag{3.5}$$

and choosing $\mathsf{L} = \mathsf{J}\,\mathsf{K}$ with K symmetric yields

$$\dot{\mathsf{P}}^{\mathsf{t}}_{t,t_o}\mathsf{J}\mathsf{P}_{t,t_o} + \mathsf{P}^{\mathsf{t}}_{t,t_o}\mathsf{J}\dot{\mathsf{P}}_{t,t_o} = 0 \quad \Rightarrow \quad \mathsf{P}^{\mathsf{t}}_{t,t_o}\mathsf{J}\mathsf{P}_{t,t_o} = e^{-\mathsf{L}^{\mathsf{t}}(t-t_o)}\mathsf{J}e^{-\mathsf{L}\,(t-t_o)} = \mathsf{J} \tag{3.6}$$

□

Proposition 3.2. *There exists a change of variables mapping (2.1) with* H_t *periodic into*

$$\dot{\boldsymbol{\xi}}_t = \mathsf{J}\mathsf{K}\cdot\boldsymbol{\xi}_t \quad \text{with} \quad \mathsf{K} = \mathsf{K}^{\mathsf{t}} \tag{3.7}$$

where K *is constant.*

Proof. Suppose that (2.1) holds and set

$$\boldsymbol{\chi}_t = \mathsf{P}_{t,t_o}\cdot\boldsymbol{\xi}_t \tag{3.8}$$

Taking the time derivative, we obtain

$$\begin{aligned}\dot{\boldsymbol{\chi}}_t = \frac{d}{dt}\mathsf{P}_{t,t_o}\cdot\boldsymbol{\xi}_t = \\ \dot{\mathsf{F}}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\cdot\boldsymbol{\xi}_t - \mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\mathsf{J}\,\mathsf{K}\cdot\boldsymbol{\xi}_t + \mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\cdot\dot{\boldsymbol{\xi}}_t = \mathsf{J}\,\mathsf{H}_t\cdot\boldsymbol{\chi}_t\end{aligned} \tag{3.9}$$

where the last equality must hold by hypothesis. Then, we observe that

$$\dot{\mathsf{F}}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\cdot\boldsymbol{\xi}_t = \mathsf{J}\mathsf{H}_t\mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\cdot\boldsymbol{\xi}_t = \mathsf{J}\mathsf{H}_t\mathsf{P}_{t,t_o}\cdot\boldsymbol{\xi}_t = \mathsf{J}\mathsf{H}_t\cdot\boldsymbol{\chi}_t \tag{3.10}$$

hence we must also have

$$0 = -\mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\mathsf{J}\,\mathsf{K}\cdot\boldsymbol{\xi}_t + \mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\cdot\dot{\boldsymbol{\xi}}_t = -\mathsf{F}_{t,t_o}\,e^{-\mathsf{J}\,\mathsf{K}\,(t-t_o)}\left(\mathsf{J}\,\mathsf{K}\cdot\boldsymbol{\xi}_t - \dot{\boldsymbol{\xi}}_t\right) \tag{3.11}$$

□

Proposition 3.3 (Gel'fand-Lidskii)**.** *Every curve* $\mathsf{F}(t)$ *in* $Sp(2n)$ $t\in[0,T]$ *with step continuous derivatives satisfies an equation of the form*

$$\dot{\mathsf{F}} = \mathsf{J}\,\mathsf{H}\,\mathsf{F}_t\,, \qquad \mathsf{H} = \mathsf{H}^{\mathsf{t}} \tag{3.12}$$

Proof.

$$0 = \frac{d}{dt}\mathsf{F}^{\mathsf{t}}_t\,\mathsf{J}\,\mathsf{F}_t \qquad \Rightarrow \qquad \mathsf{J}\dot{\mathsf{F}}_t = -{F^{\mathsf{t}}_t}^{-1}\dot{\mathsf{F}}^{\mathsf{t}}_t\,\mathsf{J}\,\mathsf{F}_t \tag{3.13}$$

Define

$$\mathsf{H}_t = {\mathsf{F}^{\mathsf{t}}_t}^{-1}\dot{\mathsf{F}}^{\mathsf{t}}_t\mathsf{J} = {\mathsf{F}^{\mathsf{t}}}^{-1}\mathsf{J}\dot{\mathsf{F}}^{-1}_t \tag{3.14}$$

it is enough to prove that it is symmetric

$$\mathsf{H}^{\mathsf{t}} = \mathsf{J}^{\mathsf{t}}\,\dot{\mathsf{F}}_t\,\mathsf{F}^{-1}_t = -\mathsf{J}^{\mathsf{t}}\,\mathsf{F}_t\,\dot{\mathsf{F}}^{-1}_t = \mathsf{J}^{\mathsf{t}}\,\mathsf{F}_t\,\mathsf{J}^2\dot{\mathsf{F}}^{-1}_t = {\mathsf{F}^{\mathsf{t}}_t}^{-1}\mathsf{J}\,\dot{\mathsf{F}}^{-1}_t \tag{3.15}$$

□

3.1 Stability of linear periodic Hamiltonian systems

Definition 3.1. *Let (2.1) be periodic with period T. The flow defined by the fundamental solution F is*

- *stable if all solutions are bounded as t tends to infinity*
- strongly *(structurally) stable if it is stable and there exists an $\epsilon > 0$ such that all $\tilde{\mathsf{H}}_t$ symmetric and periodic of period T such that*

$$\left\| \mathsf{H} - \tilde{\mathsf{H}} \right\| < \epsilon \tag{3.16}$$

have stable fundamental solution.

Remark 3.1. Since $\mathsf{F}_{t+T,0} = \mathsf{F}_{t+T,T} F_{T,0} = \mathsf{F}_{t,0} F_{T,0}$ it follows that for any positive integer n

$$\mathsf{F}_{t+n\,T,0} = \mathsf{F}_{t,0} \mathsf{F}_{T,0}^{n} \tag{3.17}$$

The necessary and sufficient condition for the stability of the flow is that $\mathsf{F}_{T,0}^{n}$ is bounded as n tends to infinity.

Definition 3.2 (Monodromy)**.** *Let (2.1) be periodic of period T. Let F be the fundamental solution. The matrix*

$$\mathsf{F}_{T,0} = \mathsf{M} \tag{3.18}$$

is called the monodromy matrix.

Proposition 3.4. *Necessary and sufficient condition for stability of the fundamental solution of a linear Hamiltonian system is that M is diagonalizable with eigenvalues on the unit circle.*

Definition 3.3 (Elliptic eigenvalues)**.** *If $\theta > 0$ the elliptic eigenvalues are referred to as*

- $\lambda = e^{\imath\,\theta}$ *first kind*
- $\lambda = e^{-\imath\,\theta}$ *second kind*

The characterisation is intrinsic. Namely the normal form of an elliptic block is

$$\mathsf{M}_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad\qquad \mathsf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.19}$$

meaning that

$$\frac{d\mathsf{M}_\theta}{dt} = \mathsf{J}\,\mathsf{M}_\theta \tag{3.20}$$

The eigenvectors are

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ \imath \end{bmatrix} = e^{\imath\,\theta} \begin{bmatrix} 1 \\ \imath \end{bmatrix} \quad \text{with} \quad [1, -\imath]\,\mathsf{J} \begin{bmatrix} 1 \\ \imath \end{bmatrix} = 2\,\imath \tag{3.21}$$

(first kind) and

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -\imath \end{bmatrix} = e^{-\imath\theta} \begin{bmatrix} 1 \\ -\imath \end{bmatrix} \quad \text{with} \quad [1, \imath] \cdot \mathsf{J} \begin{bmatrix} 1 \\ -\imath \end{bmatrix} = -2\,\imath \tag{3.22}$$

second kind. In other words, the matrix M_θ is orthogonal

$$\mathsf{M}_\theta^{\mathsf{t}} = \mathsf{M}_\theta^{-1} \tag{3.23}$$

hence it is diagonalizable on a self-dual orthonormal basis with respect to standard scalar product over $\mathbb{C}^d$

$$\langle \boldsymbol{x}_1 \,, \boldsymbol{x}_2 \rangle = \sum_{i=1}^{d} x_{1:i}^{\star} x_{2:i} \tag{3.24}$$

Referring as $\boldsymbol{v}_i$ $i = 1, 2$ as to the normalized eigenvector of, respectively, the first and second type, the symplectic form which we will refer to as Krein invariant

$$\langle \boldsymbol{v}_i \,, \mathsf{J} \cdot \boldsymbol{v}_j \rangle = \delta_{ij} (-1)^{1+\delta_{i,1}} \imath \tag{3.25}$$

is preserved by the dynamics.

Proposition 3.5 (Krein). *Let (2.1) be periodic of period T. Suppose the monodromy matrix to have $2\,k$ distinct eigenvalues on the unit circle. If the system is perturbed as*

$$\dot{\mathsf{F}}_{t;z} = \mathsf{J}\,(\mathsf{H}_t + z\,\mathsf{Q}_t) \mathsf{F}_{t;z} \tag{3.26}$$

with $\mathsf{Q}_{t+T} = \mathsf{Q}_t \in \mathrm{End}(\mathbb{R}^{2\,n})$, positive definite and $z \in \mathbb{C}$, $Im\, z > 0$ then no eigenvalue remains on the unit circle.

Proof. Let $\boldsymbol{v}_{t;\theta_m} = \mathsf{F}_{t,0;z} \boldsymbol{v}_{0;\theta_m}$ such that if $\mathsf{M} := \mathsf{F}_{T,0;0} \equiv \mathsf{F}_{T,0}$ is the unperturbed (i.e. for $z = 0$) monodromy

$$\mathsf{M} \cdot \boldsymbol{v}_{0;\theta_m} = e^{\imath \theta_m} \boldsymbol{v}_{0;\theta_m} \tag{3.27}$$

Since $\boldsymbol{v}_{t;\theta_m}$ is a solution of the dynamics for finite z

$$\langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle = \langle \boldsymbol{v}_{t;\theta_m} \,, (\mathsf{H}_t + z\,\mathsf{Q}_t) \cdot \boldsymbol{v}_{t;\theta_m} \rangle \tag{3.28}$$

and

$$\Im \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle := \frac{1}{2\,\imath} (\langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle - \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle^{*}) = \Im \langle \boldsymbol{v}_{t;\theta_m} \,, z\,\mathsf{Q}_t \cdot \boldsymbol{v}_{t;\theta_m} \rangle \tag{3.29}$$

where the last equality follows from

$$\Im \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{H} \cdot \boldsymbol{v}_{t;\theta_m} \rangle = 0 \tag{3.30}$$

On the other hand

$$\begin{aligned} &\langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle - \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \cdot \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle^{*} \\ &= \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle - \langle \mathsf{J} \cdot \boldsymbol{v}_{t;\theta_m} \,, \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle^{*} \\ &= \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}} \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \rangle - \langle \frac{d}{dt} \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J} \cdot \boldsymbol{v}_{t;\theta_m} \rangle = \frac{d}{dt} \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}}\, \boldsymbol{v}_{t;\theta_m} \rangle \end{aligned} \tag{3.31}$$

owing to $\mathsf{J} = -\mathsf{J}^{\mathsf{t}}$. We arrived to

$$\frac{d}{dt} \langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}}\, \boldsymbol{v}_{t;\theta_m} \rangle = 2\,\imath\, \Im \langle \boldsymbol{v}_{t;\theta_m} \,, z\,\mathsf{Q}_t \cdot \boldsymbol{v}_{t;\theta_m} \rangle \neq 0 \tag{3.32}$$

which shows that $\langle \boldsymbol{v}_{t;\theta_m} \,, \mathsf{J}^{\mathsf{t}}\, \boldsymbol{v}_{t;\theta_m} \rangle$ cannot be a Krein invariant at finite z. □

The theorem extends trivially to the degenerate case.

The theorem can be used to discriminate between elliptic eigenvalues of the first and of the second kind. Namely we can show that if $Sp\mathsf{Q}_t > 0$ and $Imz > 0$

$$\theta(z) = \theta(0) + z\frac{d\theta}{dz}(0) + O(z^2) := \theta(0) + z\,\theta'(0) + O(z^2) \tag{3.33}$$

then $\theta'(0) < 0$. As a consequence

$$e^{\imath\theta(z)} = e^{\imath[\theta(0)+\Re\{z\}\theta'(0)]-\Im\{z\}\theta'(0)} \tag{3.34}$$

moves *outside* the unit circle.

$$e^{-\imath\theta(z)} = e^{-\imath[\theta(0)+\Re\{z\}\theta'(0)]+\Im\{z\}\theta'(0)} \tag{3.35}$$

moves *inside* the unit circle. To prove the claim we can use a perturbative argument. Suppose $|z| \ll 1$. Then

$$\mathsf{M}_z = \mathsf{M} + z\int_0^T dt\,\mathsf{M}\,\mathsf{F}_{t,0}^{-1}\mathsf{J}\,\mathsf{Q}_t\,\mathsf{F}_{t,0} \tag{3.36}$$

projecting on the unperturbed eigenvector

$$\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}_z\cdot\boldsymbol{v}_{\theta_m}\rangle = \langle \boldsymbol{v}_{\theta_m}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}\cdot\boldsymbol{v}_{\theta_m}\rangle + z\int_0^T dt\,\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}\mathsf{F}_{t,0}^{-1}\,\mathsf{J}\,\mathsf{Q}_t\,\mathsf{F}_{t,0}\cdot\boldsymbol{v}_{\theta_m}\rangle \tag{3.37}$$

whence

$$\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}_z\cdot\boldsymbol{v}_{\theta_m}\rangle = 2\,\imath\,e^{\imath\theta_m} + z\int_0^T dt\,\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{M}^{-1\,\mathsf{t}}\mathsf{J}^{\mathsf{t}}\mathsf{F}_{t,0}^{-1}\mathsf{J}\,\mathsf{Q}_t\,\mathsf{F}_{t,0}\boldsymbol{v}_{\theta_m}\rangle \tag{3.38}$$

The latter equality follows from

$$\mathsf{M}^{\mathsf{t}}\,\mathsf{J}\,\mathsf{M} = \mathsf{J} \tag{3.39}$$

Recalling that

$$\mathsf{M}^{\mathsf{t}}\cdot\boldsymbol{v}_{\theta_m} = \mathsf{M}^{-1}\cdot\boldsymbol{v}_{\theta_m} = e^{-\imath\theta_m}\boldsymbol{v}_{\theta_m} \tag{3.40}$$

we obtain

$$\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}\cdot\boldsymbol{v}_{\theta_m}\rangle = 2\,\imath\,e^{\imath\theta_m}\left[1 + \frac{z}{2\,\imath}\int_0^T dt\,\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{F}_{t,0}^{\mathsf{t}}\,\mathsf{Q}_t\,\mathsf{F}_{t,0}\cdot\boldsymbol{v}_{\theta_m}\rangle\right] \tag{3.41}$$

On the other hand if

$$\langle \boldsymbol{v}_{\theta_m(z)}\,, \mathsf{J}^{\mathsf{t}}\mathsf{M}\cdot\boldsymbol{v}_{\theta_m(z)}\rangle = e^{\imath\,\theta_m(z)}2\,\imath \approx\, e^{\imath\,\theta_m(0)}2\,\imath[1+\imath\,z\,\theta'(0)] + O(z^2) \tag{3.42}$$

by comparison

$$\theta'(0) = -\frac{1}{2}\int_0^T dt\,\langle \boldsymbol{v}_{\theta_m}\,, \mathsf{F}_{t,0}^{\mathsf{t}}\,\mathsf{Q}_t\,\mathsf{F}_{t,0}\cdot\boldsymbol{v}_{\theta_m}\rangle\,<\,0 \tag{3.43}$$

Theorem 3.1 (Krein)**.** *If a stable linear Hamiltonian system does not have degenerate eigenvalues of different kind the system is strongly stable*

Idea of the proof: The calculation above shows that elliptic eigenvalues of different kind exit the unit circle in different ways. The result would be a dislocation of the eigenvalues, incompatible with the generation of a loxodromic quartet.

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