Lecture 03: Non-autonomous linear systems

1 Non homogeneous linear systems

Let $\mathsf{A} \in \mathrm{End}(\mathbb{R}^n)$, consider the initial data problem

$$\begin{aligned}\dot{\boldsymbol{\phi}}_t &= \mathsf{A}\,\boldsymbol{\phi}_t + \boldsymbol{f}_t \\ \boldsymbol{\phi}_0 &= \boldsymbol{x}_o\end{aligned} \tag{1.1}$$

Proposition 1.1. *The unique solution of (1.1) is*

$$\boldsymbol{\phi}_t = e^{\mathsf{A}\,t}\,\boldsymbol{x}_o + \int_0^t ds\, e^{\mathsf{A}\,(t-s)}\,\boldsymbol{f}_s \tag{1.2}$$

Proof. The solution is obtained using the method of *variation of constants*:

$$\boldsymbol{\phi}_t = e^{\mathsf{A}\,t}\,\boldsymbol{\psi}_t \tag{1.3}$$

then

$$e^{\mathsf{A}\,t}\,\dot{\boldsymbol{\psi}}_t = \boldsymbol{f}_t \qquad \Rightarrow \qquad \dot{\boldsymbol{\psi}}_t = e^{-\mathsf{A}\,t}\,\boldsymbol{f}_t \tag{1.4}$$

thus

$$\boldsymbol{\phi}_t = e^{\mathsf{A}\,t}\left(\boldsymbol{x}_o + \int_0^t ds\, e^{-\mathsf{A}\,s}\,\boldsymbol{f}_s\right) \tag{1.5}$$

□

Remark 1.1. The initial conditions are stored in the homogeneous part of the solution!!!

2 Linear non-autonomous dynamics

We consider the linear ordinary differential equation

$$\dot{\boldsymbol{\phi}}_t = \mathsf{A}_t \cdot \boldsymbol{\phi}_t \tag{2.1a}$$

$$\boldsymbol{\phi}_{t_o} = \boldsymbol{x} \tag{2.1b}$$

We suppose the vector field $\boldsymbol{f}(t, \boldsymbol{x}) = \mathsf{A}_t \cdot \boldsymbol{x}$ driving (2.1a) to be smooth also in its time dependence

$$\boldsymbol{f} \in \mathrm{C}^r(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n) \tag{2.2}$$

Hence we can express the solution of (2.1) in terms of the linear flow describing the fundamental solution of (2.1a)

$$\boldsymbol{\phi}_t = \mathsf{F}_{t,t_o} \cdot \boldsymbol{x} \tag{2.3}$$

Using "Picard"-iterations of (2.1a), the flow F is obtained as a *time ordered* exponential

$$\mathsf{F}_{t,t_o} = 1 + \sum_{n>0} \int_{t_o}^{t} dt_n \int_{t_o}^{t_n} dt_{n-1} ... \int_{t_o}^{t_3} dt_2 \int_{t_o}^{t_2} dt_1 \, \mathsf{A}_{t_n} \, \mathsf{A}_{t_{n-1}} \dots \mathsf{A}_{t_2} \, \mathsf{A}_{t_1} \; := \; \mathcal{T} \left\{ e^{\int_{t_0}^{t} ds \mathsf{A}_s} \right\} \tag{2.4}$$

If

$$[\mathsf{A}_t, \mathsf{A}_s] = 0 \qquad \forall \, t, s \quad \Rightarrow \quad \mathsf{F}_{t,t_o} = e^{\int_{t_o}^{t} ds \, \mathsf{A}_s} \tag{2.5}$$

in particular, the time-order exponential reduces to the ordinary one if the matrix A is autonomous.

Remark 2.1.

$$\mathsf{F}_{t,0} = \mathsf{F}_{t,t_o} \mathsf{F}_{t_o,0} \qquad \Rightarrow \qquad \mathsf{F}_{t,t_o} = \mathsf{F}_{t,0} \mathsf{F}_{t_o,0}^{-1} \tag{2.6}$$

thus also in the non-autonomous case it is enough to know $F_{t,0}$ in order to reconstruct the flow for other initial times.

Proposition 2.1. *The solution of (2.1) is unique*

Proof. Suppose there exist two solutions $\phi_{t;1}$ and $\phi_{t;2}$

$$\phi_{t;1} = \phi_{t_o;1} + \int_{t_o}^{t} ds \, \mathsf{A}_s \cdot \phi_{s;1} \qquad i = 1, 2 \tag{2.7}$$

Then

$$\| \, \phi_{t;1} - \phi_{t;2} \, \| \leq \int_{t_o}^{t} ds \; \| \, \mathsf{A}_s \, \| \, \| \, \phi_{s;1} - \phi_{s;2} \, \| \leq a \int_{0}^{t} ds \; \| \, \phi_{s;1} - \phi_{s;2} \, \| \tag{2.8}$$

where

$$a = \sup_{s,i,j} |A_{ij}(s)| \tag{2.9a}$$

whence the claim. □

2.1 Periodic case

Consider

$$\dot{\mathsf{F}}_{t,t_o} = \mathsf{A}_t \, \mathsf{F}_{t,t_o} \, , \qquad \mathsf{A}_{t+T} \; = \; \mathsf{A}_t \tag{2.10a}$$

$$\lim_{t \downarrow t_o} \mathsf{F}_{t,t_o} = 1 \tag{2.10b}$$

Theorem 2.1 (Floquet)**.** *Let $\mathsf{A}_t = \mathsf{A}_{t+T}$ for all t. Then the principal solution of (2.1a) can be written as*

$$\mathsf{F}_{t,t_o} = \mathsf{P}_{t,t_o} e^{\mathsf{B}\,(t-t_o)} \tag{2.11}$$

where

$$\begin{aligned} &\mathsf{P}_{t+T\,t_o} = \mathsf{P}_{t t_o} \qquad \forall \; t \\ &\mathsf{P}_{t_o\,t_o} = 1 \end{aligned} \tag{2.12}$$

and B is a constant matrix.

Proof. The proof proceed in three steps

1. We first prove that

$$\mathsf{F}_{t+T,t_o} = \mathsf{F}_{t,t_o}\,\mathsf{F}_{t_o+T,t_o} \tag{2.13}$$

To this goal let us define the auxiliary matrix

$$\mathsf{G}_{t,t_o} := \mathsf{F}_{t+T,t_o} = \mathsf{F}_{t+T,t_o+T}\mathsf{F}_{t_o+T,t_o} \tag{2.14}$$

We see that it satisfies

$$\dot{\mathsf{G}}_{t,t_o} \;=\; \dot{\mathsf{F}}_{t+T,t_o} \;=\; \mathsf{A}_{t+T}\,\mathsf{F}_{t+T,t_o} = \mathsf{A}_t\,\mathsf{G}_{t,t_o} \tag{2.15}$$

which is the same as (2.10a), with boundary condition

$$\mathsf{G}_{t_o,t_o} := \mathsf{F}_{t_o+T,t_o} \tag{2.16}$$

Liouville theorem guarantees that

$$\det \mathsf{F}_{t,t_o} \;\neq\; 0 \qquad \forall\, t \tag{2.17}$$

hence F_{t,t_o} is invertible. We can therefore construct the flow

$$\tilde{\mathsf{F}}_{t,t_o} = \mathsf{G}_{t,t_o}\mathsf{F}^{-1}_{t_o+T,t_o} \tag{2.18}$$

which now satisfies both (2.10a), and (2.10a) including the initial condition. Since the hypotheses of the theorems of existence and uniqueness hold true, we must then have

$$\mathsf{G}_{t,t_o}\mathsf{F}^{-1}_{t_o+T,t_o} \equiv \mathsf{F}_{t+T,t_o+T} \;=\; \mathsf{F}_{t,t_o} \qquad \Rightarrow \qquad \mathsf{F}_{t+T,t_o} = \mathsf{F}_{t,t_o}\,\mathsf{F}_{t_o+T,t_o} \tag{2.19}$$

2. We now claim that there exists a matrix B such that

$$\mathsf{F}_{t_o+T,t_o} = e^{\mathsf{B}\,T} \tag{2.20}$$

To see this, let $\lambda_i \neq 0$ and m_i, $i = 1,\dots,m$ be respectively the eigenvalues of F_{t_o+T,t_o} and their algebraic multiplicities. Let

$$\mathsf{F}_{t_o+T,t_o} = \sum_{i=1}^{m}(\lambda_i\mathsf{P}_i + \mathsf{N}_i) \tag{2.21}$$

be the decomposition of F_{t_o+T,t_o} into its semi-simple and nilpotent parts. Here P_i is the projector on the subspace spanned by the generalized eigenvectors associated to the eigenvalue λ_i and N_i is the nilpotent component of F acting on that subspace. We recall that

$$\mathsf{P}_i\mathsf{P}_j = \mathsf{P}_j\mathsf{P}_i = \delta_{ij}\mathsf{P}_i \tag{2.22}$$

and

$$\mathsf{N}_i\mathsf{N}_j = \mathsf{N}_j\mathsf{N}_i = \delta_{ij}\mathsf{N}_i^2 \tag{2.23}$$

and that $[\mathsf{N}_i, \mathsf{P}_j] = 0$. It follows that

$$\mathsf{B} = \frac{1}{T}\sum_{i=1}^{m}\left[\ln(\lambda_i)\,\mathsf{P}_i - \sum_{j=1}^{m_i}\frac{(-\,\mathsf{N}_i)^j}{j\,\lambda_i^j}\right] \tag{2.24}$$

satisfies

$$e^{\mathsf{B}\,T} = \mathsf{F}_{t_o+T,t_o} \tag{2.25}$$

Namely the chain of equality

$$\ln\left\{\sum_{i=1}^{m}(\lambda_i\,\mathsf{P}_i + \mathsf{N}_i)\right\} = \ln\left\{\sum_{i=1}^{m}\lambda_i\mathsf{P}_i\left(1+\frac{\mathsf{N}_i}{\lambda_i}\right)\right\} = \sum_{i=1}^{m}\left\{\ln(\lambda_i)\,\mathsf{P}_i + \ln\left(1+\frac{\mathsf{N}_i}{\lambda_i}\right)\right\} \tag{2.26}$$

holds true owing to the

(2.22) and (2.23). The matrix B so determined is unique modulo a phase associated to the winding number of the imaginary part of the logarithms.

3. The third step consists in verifying that

$$\mathsf{P}_{tt_o} = \mathsf{F}_{t,t_o}e^{-\mathsf{B}\,(t-t_o)}. \tag{2.27}$$

defines a periodic matrix. Namely we see that for all $\forall\, t\ \in\ \mathbb{R}$

$$\mathsf{P}_{t+T,t_o} = \mathsf{F}_{t+T,t_o}\,e^{-\mathsf{B}(t+T-t_o)} = \mathsf{F}_{t,t_o}\,e^{\mathsf{B}\,T}\,e^{-B(t+T-t_o)} = \mathsf{P}_{t,t_o} \tag{2.28}$$

Finally, it is straightforward to check that

$$\mathsf{P}_{t_o,t_o} = \mathsf{F}_{t_o,t_o} = 1 \tag{2.29}$$

which completes the proof.

□

Floquet's theorem shows that the solution of (2.10) for $t_o = 0$ can be written as

$$\boldsymbol{x}_t = \mathsf{P}_{t0}\,e^{\mathsf{B}\,t}\,\boldsymbol{x} \tag{2.30}$$

Since P_{t,t_o} is periodic, the long-time behavior depends only on B. The eigenvalues of B are called the *characteristic exponents* of the equation.

Definition 2.1 (monodromy)**.** *The matrix*

$$\mathsf{M} = e^{\mathsf{B}\,T} \tag{2.31}$$

defined by Floquet's theorem is called the **monodromy matrix**.

The eigenvalues of the monodromy matrix, called the *characteristic multipliers*, are exponentials of the characteristic exponents times T. Computing the characteristic exponents is difficult in general, but the existence of the representation (2.30) is already useful to classify the possible behaviors near a periodic orbit.

Example 2.1. Let us illustrate how formula (2.24) comes about in an elementary example

$$\mathsf{B} = \ln\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} = \ln\lambda + \ln\begin{bmatrix}1 & \frac{1}{\lambda}\\ 0 & 1\end{bmatrix} \tag{2.32}$$

Since

$$\ln(1+x) = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n} \tag{2.33}$$

then

$$\ln\begin{bmatrix}1 & \frac{1}{\lambda}\\ 0 & 1\end{bmatrix} := -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\begin{bmatrix}0 & \frac{1}{\lambda}\\ 0 & 0\end{bmatrix}^n = \begin{bmatrix}0 & \frac{1}{\lambda}\\ 0 & 0\end{bmatrix} \tag{2.34}$$

owing to nilpotence.

2.2 Linear perturbation theory

Consider a family of linear ODE's

$$\begin{aligned} \dot{\boldsymbol{x}}_{t;\mu} &= \mathsf{A}_{t:\mu}\boldsymbol{x}_{t;\mu} \\ \dot{\boldsymbol{x}}_{0;\mu} &= \boldsymbol{x}_o \end{aligned} \tag{2.35}$$

where $\mathsf{A} \in \mathrm{End}(\mathbb{R}^n)$ smoothly depends on μ. We want to study the solution in a neighborhood of some μ_0. Let

$$\begin{aligned} \boldsymbol{x}'_t &:= \partial_\mu \boldsymbol{x}_{t;\mu}|_{\mu=\mu_0} \\ \mathsf{A}'_t &= \partial_\mu \mathsf{A}_{t;\mu}|_{\mu=\mu_0} \end{aligned} \tag{2.36}$$

The equation for the variation is

$$\dot{\boldsymbol{x}}' = \mathsf{A}_{t:\mu_o} \cdot \boldsymbol{x}'_t + \mathsf{A}'_{t:\mu_o}\boldsymbol{x}_{t;\mu_0} \tag{2.37}$$

the solution is

$$\boldsymbol{x}'_t = \int_0^t ds\, \mathsf{F}_{t\,0:\mu_o}\, \mathsf{F}^{-1}_{s\,0:\mu_o} \mathsf{A}'_{s:\mu_o} \cdot \boldsymbol{x}_{s;\mu_0} \tag{2.38}$$

since

$$\boldsymbol{x}'_0 = 0 \tag{2.39}$$