Lecture 02: Notes on linear systems

Introduction

The expounded material can be found in

- ch. 2 of [2]
- ch. 2 of [1]

1 Some terminology and mathematical notation for linear maps

Let $\mathbb{V}_1$, $\mathbb{V}_2$ two vector spaces.

Definition 1.1. *The map*

$$\mathsf{A}\colon \mathbb{V}_1 \mapsto \mathbb{V}_2 \tag{1.1}$$

is linear if

- *for any* $\boldsymbol{x}_1\,, \boldsymbol{x}_2 \in V_1$, *is* additive

$$\mathsf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \mathsf{A}\boldsymbol{x}_1 + \mathsf{A}\boldsymbol{x}_2$$

- *for any* $c \in \mathbb{C}$ *and any* $\boldsymbol{x} \in V_1$

$$\mathsf{A}\, c\, \boldsymbol{x} = c\, \mathsf{A}\boldsymbol{x} \tag{1.2}$$

By definition a linear map preserves the algebraic structure of its domain, a vector space, as its image is still a vector space. It is therefore an *homomorphism*. The collection of linear maps between the two vector spaces $\mathbb{V}_1$, $\mathbb{V}_2$ can be denoted as $\mathrm{Hom}(\mathbb{V}_1, \mathbb{V}_2)$. In particular, if $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}$, we can write $\mathrm{Hom}(\mathbb{V}, \mathbb{V}) = \mathrm{End}(\mathbb{V})$ and talk of the set of *endomorphisms* of $\mathbb{V}$. If $\mathsf{A} \in \mathrm{Hom}(\mathbb{V}_1, \mathbb{V}_2)$ is *invertible* we can think of it as an element of $\mathrm{GL}(\mathbb{V}_1, \mathbb{V}_2)$ the general linear group. Finally, in any basis of $\mathbb{V} = \mathbb{R}^n$, A is a matrix i.e. $\mathsf{A} \in \mathbb{R}^{n^2}$. In what follow we will identify the map with its coordinate representation.

2 Linear differential equations

Let $A \in \mathrm{End}(\mathbb{R}^n)$, we consider

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \mathsf{A}\, \boldsymbol{x}(t) \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0 \end{aligned} \tag{2.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$. Formally (2.1) has the solution

$$\boldsymbol{x}(t) = e^{\mathsf{A}\, t}\, \boldsymbol{x}_0 \tag{2.2}$$

2.1 Exponential of matrices

Definition 2.1 (exponential of a matrix)**.** *Let* $A \in \mathrm{End}(\mathbb{R}^n)$*, then*

$$e^{\mathsf{A}} := \sum_{n=0}^{\infty} \frac{\mathsf{A}^n}{n!} \tag{2.3}$$

Remark 2.1. The norm over $\mathrm{End}(\mathbb{R}^n)$ defined by

$$\| \mathsf{A} \| = \sup_{\boldsymbol{x} \in \mathbb{R}^n} \{ \| \mathsf{A}\boldsymbol{x} \| \ | \ \| \boldsymbol{x} \| = 1 \} \leq \left(\sum_{i\,j=1}^{n} \mathsf{A}_{ij}^2 \right)^{1/2} \tag{2.4}$$

verifies

$$\| \mathsf{A}\,\mathsf{B} \| \leq \| \mathsf{A} \| \, \| \mathsf{B} \| \tag{2.5}$$

whence $\|A^n\| \leq \|A\|^n$, thus the series defining e^A is convergent. To prove the claim we can use the identity

$$\frac{\langle \mathsf{A}\,\mathsf{B}\,\boldsymbol{x}\,, \mathsf{A}\,\mathsf{B}\,\boldsymbol{x} \rangle}{\langle \boldsymbol{x}\,, \boldsymbol{x} \rangle} = \frac{\langle \mathsf{AB}\boldsymbol{x}\,, \mathsf{AB}\boldsymbol{x} \rangle}{\langle \mathsf{B}\boldsymbol{x}\,, \mathsf{B}\boldsymbol{x} \rangle} \frac{\langle \mathsf{B}\boldsymbol{x}\,, \mathsf{B}\boldsymbol{x} \rangle}{\langle \boldsymbol{x}\,, \boldsymbol{x} \rangle} \tag{2.6}$$

Proposition 2.1. *Let* $\mathsf{A}\,, \mathsf{B}$ *and* T*. Then*

1. *if* $\mathsf{B} = \mathsf{T}\,\mathsf{A}\,\mathsf{T}^{-1}$ *i.e. if* A *and* B *are similar then* $\exp(\mathsf{B}) = \mathsf{T} \exp(\mathsf{A}) \mathsf{T}^{-1}$*;*
2. *if* $[\mathsf{A}\,, \mathsf{B}]$ *then* $\exp(\mathsf{A} + \mathsf{B}) = \exp(\mathsf{A}) \exp(\mathsf{B})$*;*
3. $\exp(-\mathsf{A}) = (\exp(\mathsf{A}))^{-1}$

Proof.

1. If A and B are similar

$$\exp(\mathsf{B}) = \sum_{n=0}^{\infty} \frac{(\mathsf{T}\,\mathsf{A}\,\mathsf{T}^{-1})^n}{n!} = \sum_{n=0}^{\infty} \mathsf{T} \frac{\mathsf{A}^n}{n!} \mathsf{T}^{-1} \tag{2.7}$$

2. If A and B commute

$$\exp(\mathsf{A} + \mathsf{B}) = \sum_{n=0}^{\infty} \frac{(\mathsf{A} + \mathsf{B})^n}{n!} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{\mathsf{A}^j}{j!} \frac{\mathsf{B}^k}{k!} \tag{2.8}$$

   whence (omitting considerations about convergence of the series)

$$\sum_{n=0}^{\infty} \sum_{j+k=n} \frac{\mathsf{A}^j}{j!} \frac{\mathsf{B}^k}{k!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\mathsf{A}^j}{j!} \frac{\mathsf{B}^{n-j}}{(n-j)!} = \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{\mathsf{A}^j}{j!} \frac{\mathsf{B}^{n-j}}{(n-j)!} = \sum_{j=0}^{\infty} \frac{\mathsf{A}^j}{j!} \sum_{k=0}^{\infty} \frac{\mathsf{B}^k}{k!} \tag{2.9}$$

3. Set $\mathsf{B} = -\mathsf{A}$ and use the property *2*.

□

Proposition 2.2. *If* $\boldsymbol{v}_{[a]} \in \mathbb{R}^n$ *is an eigenvector of* A *with eigenvalue* a *is also an eigenvector of* $\exp(\mathsf{A})$ *with eigenvalue* $\exp(a)$

Proposition 2.3. *Let* $\mathsf{A} \in \mathrm{End}(\mathbb{R}^d)$

$$\frac{d}{d\,t} e^{\mathsf{A}t} = \mathsf{A}\, e^{\mathsf{A}\,t} = e^{\mathsf{A}\,t}\,\mathsf{A} \tag{2.10}$$

Proof.

$$\frac{d}{d\,t} e^{\mathsf{A}\,t} = \lim_{\epsilon \downarrow 0} \frac{e^{\mathsf{A}\,t}\left(e^{\mathsf{A}\,\varepsilon} - 1\right)}{\varepsilon} = \mathsf{A}\, e^{\mathsf{A}\,t} \tag{2.11}$$

□

2.2 Existence and uniqueness of solutions

Existence and uniqueness follow from the more general Peano-Cauchy and icard-Lindelöf theorems. It is, however, instructive to prove them in the linear case.

Theorem 2.1. *The solution of (2.1) exists and is unique.*

Proof.

- Existence follows from proposition 2.3.
- Uniqueness: the proof proceed per absurdum. Let us surmise that

$$\boldsymbol{z}(t) = e^{-\mathsf{A}\,t}\boldsymbol{x}(t) \tag{2.12}$$

is *not* constant. Then, omitting the time dependence to neaten the notation,

$$\dot{\boldsymbol{z}} = -\mathsf{A}\, e^{-\mathsf{A}t}\,\boldsymbol{x} + e^{-\mathsf{A}t}\dot{\boldsymbol{x}} = -\mathsf{A}\, e^{-\mathsf{A}t}\,\boldsymbol{x} + e^{-\mathsf{A}t}\,\mathsf{A}\,\boldsymbol{x} = 0 \tag{2.13}$$

which contradicts the hypothesis.

□

2.3 Explicit form of the solution

Definition 2.2. *The fundamental (principal) solution* F_t *of (2.1) is specified by the initial condition*

$$\mathsf{F}_0 = 1 \qquad \Rightarrow \qquad \mathsf{F}_t = e^{\mathsf{A}\,t} \tag{2.14}$$

Proposition 2.4. F_t *is a flow*

The characteristic polynomial of $\mathsf{A} \in \mathrm{End}(\mathbb{R}^n)$ can be written as

$$P(\mathsf{A}) = \det(\lambda\,1 - \mathsf{A}) = \prod_{j=1}^{m} (\lambda - a_j)^{m_j}, \tag{2.15}$$

where

- $a_1, \ldots, a_m \in \mathbb{C}$ are the $m \leq n$ distinct eigenvalues of A;

- m_j is the *algebraic multiplicity* of the eigenvalue a_j.

By definition one has

$$\sum_{j=1}^{m} m_j = n \tag{2.16}$$

The *geometric multiplicity* g_j of a_j is defined as the number of independent eigenvectors associated with a_j, and satisfies $1 \geq g_j \geq m_j$.

Proposition 2.5. *Any* $\mathsf{A} \in \mathrm{End}(\mathbb{R}^n)$ *can be decomposed as*

$$\mathsf{A} = \mathsf{S} + \mathsf{N}, \qquad \mathsf{S}\,\mathsf{N} = \mathsf{N}\,\mathsf{S}. \tag{2.17}$$

where the linear maps S, N *are defined as follows.*

- S *is the* semi-simple *part. It can be written as*

$$\mathsf{S} = \sum_{j=1}^{m} a_j\,\mathsf{P}_j, \tag{2.18}$$

where the P_j *are projectors on the eigenspaces of* A*, i.e.* P_j *is the projector over the linear subspace*

$$\mathcal{A}_j = \{\boldsymbol{v} \in \mathbb{R}^n \mid (\mathsf{A} - a_j\,1)^{m_j}\boldsymbol{v} = 0\} \tag{2.19}$$

The projectors $\{\mathsf{P}_j\}_{j=1}^{m}$ *satisfy the orthogonality relations*

$$\mathsf{P}_j\,\mathsf{P}_k = \delta_{j\,k}\,\mathsf{P}_j \qquad \sum_j \mathsf{P}_j = 1 \tag{2.20}$$

and

$$m_j = \dim(\mathsf{P}_j\,\mathbb{R}^n) \tag{2.21}$$

- *The nilpotent part* N *can be written as*

$$\mathsf{N} = \sum_{j=1}^{m} \mathsf{N}_j, \tag{2.22}$$

Each of the N_j*'s satisfy the relations*

$$\mathsf{N}_j^{m_j} = 0, \qquad \mathsf{N}_j\,\mathsf{N}_k = 0 \quad \textit{for } j \neq k, \qquad \mathsf{P}_j\,\mathsf{N}_k = \mathsf{N}_k\,\mathsf{P}_j = \delta_{jk}\,\mathsf{N}_j. \tag{2.23}$$

In an appropriate basis, each N_j is block-diagonal, with g_j blocks of the form

$$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \tag{2.24}$$

In fact, $\mathsf{N}_j = 0$ unless $g_j < m_j$. In other words,

- if for each eigenvalue the geometric multiplicity coincides with the algebraic multiplicity we can write

$$\mathsf{A} = \sum_{i=1}^{m} a_i \sum_{j=1}^{m_j} \boldsymbol{l}_{[a_i]_j} \otimes \boldsymbol{r}_{[a_i]_j} \tag{2.25}$$

with

$$\mathsf{A}\, \boldsymbol{l}_{[a_1]} = a_i\, \boldsymbol{l}_{[a_1]} \qquad \& \qquad \boldsymbol{r}_{[a_1]}\, \mathsf{A} = a_i\, \boldsymbol{r}_{[a_1]} \tag{2.26}$$

and

$$\langle \boldsymbol{r}_{[a_i]_j}\, , \boldsymbol{l}_{[a_l]_k} \rangle = \boldsymbol{r}^*_{[a_i]_j} \cdot \boldsymbol{l}_{[a_l]_k} = \sum_{s=1}^{n} (\boldsymbol{r}_{[a_i]_j})^*_s \left(\boldsymbol{l}_{[a_l]_k}\right)_s = \delta_{il}\, \delta_{jk} \tag{2.27}$$

- if the geometric multiplicity is smaller than the algebraic multiplicity of the eigenvalue a_i, we have for "left" generalized eigenvectors the relations

$$\begin{aligned} &\mathsf{A}\, \boldsymbol{l}_{[a_i]_1} = a_i\, \boldsymbol{l}_{[a_i]_1} + \boldsymbol{l}_{[a_i]_2} \\ &\mathsf{A}\, \boldsymbol{l}_{[a_i]_2} = a_i\, \boldsymbol{l}_{[a_i]_2} + \boldsymbol{l}_{[a_i]_3} \\ &\dots \\ &\mathsf{A}\, \boldsymbol{l}_{[a_i]_{m_j}} = a_i\, \boldsymbol{l}_{[a_i]_{m_j}} \end{aligned} \tag{2.28}$$

  and similarly for "right" generalized eigenvectors in the dual space.

Proposition 2.6. *The fundamental solution of (2.1) is amenable to the form*

$$e^{\mathsf{A}\,t} = \sum_{j=1}^{m} e^{a_j\,t} \mathsf{P}_j \left[1 + \mathsf{N}_j\, t + \dots + \frac{\mathsf{N}_j^{m_j-1}\, t^{m_j-1}}{(m_j-1)!} \right] \tag{2.29}$$

Proof. Whenever $\mathsf{A}\,\mathsf{B} = \mathsf{B}\,\mathsf{A}$ we can use $e^{\mathsf{A}\,t} e^{\mathsf{B}\,t} = e^{(\mathsf{A}+\mathsf{B})t}$. Thus

$$e^{\mathsf{A}\,t} = e^{(\mathsf{S}+\mathsf{N})\,t} = e^{\mathsf{S}\,t}\, e^{\mathsf{N}\,t} \tag{2.30}$$

Recalling that

$$\mathsf{P}_i\, \mathsf{P}_j = \mathsf{P}_j\, \mathsf{P}_i = \delta_{ij} \mathsf{P}_i \tag{2.31}$$

i.e. commutativity and idempotence of projectors we have

$$e^{a_j\, \mathsf{P}_j\, t} = \sum_{n=0}^{\infty} \frac{a_j^n\, \mathsf{P}_j^n\, t^n}{n!} = 1 + \mathsf{P}_j \sum_{n=1}^{\infty} \frac{a_j^n\, t^n}{n!} = 1 + (e^{a_j\,t} - 1)\, \mathsf{P}_j \tag{2.32}$$

whence

$$e^{\mathsf{S}\,t} = \prod_{j=1}^{m} e^{a_j\, \mathsf{P}_j\, t} = \prod_{j=1}^{m} \left\{ 1 + (e^{a_j\,t} - 1)\, \mathsf{P}_j \right\} = 1 + \sum_{j=1}^{m} (e^{a_j t} - 1)\, \mathsf{P}_j = \sum_{j=1}^{m} e^{a_j\,t}\, \mathsf{P}_j \tag{2.33}$$

Similarly by

$$\mathsf{N}_i \mathsf{N}_j = \mathsf{N}_j \mathsf{N}_i \tag{2.34}$$

we also have

$$
\begin{aligned}
e^{\mathsf{N}\,t} &= \prod_{j=1}^{m} e^{\mathsf{N}_j\,t} = 1 + \sum_{j=1}^{m}(e^{\mathsf{N}_j\,t} - 1) \\
&= 1 + \mathsf{N}_j\,t + \dots + \frac{\mathsf{N}_j^{m_j-1}\,t^{m_j-1}}{(m_j-1)!}
\end{aligned}
\tag{2.35}
$$

since $e^{\mathsf{N}_j\,t}$ contains only finitely many terms, being nilpotent. □

The expression (2.29) shows that the long-time behaviour is determined by the real parts of the eigenvalues a_j, while the nilpotent terms, when present, influence the short time behaviour. This motivates the following terminology:

2.4 Asymptotic behavior

Definition 2.3. *The subspace*

$$
W_u := \left\{ y \in \mathbb{R}^n \mid \lim_{t\to-\infty} e^{\mathsf{A}\,t} y = 0 \right\} \qquad P^{(u)} := \sum_{j:\Re a_j > 0} P_j \tag{2.36}
$$

is referred to as the unstable *subspace of the fixed point* $x^\star = 0$

Definition 2.4. *The subspace*

$$
W_s := \left\{ y \in \mathbb{R}^n \mid \lim_{t\to\infty} e^{\mathsf{A}\,t} y = 0 \right\} \qquad P^{(s)} := \sum_{j:\Re a_j < 0} P_j \tag{2.37}
$$

is referred to as the stable *subspace of the fixed point* $x^\star = 0$

Definition 2.5. *The subspace*

$$
W_0 := P^{(c)}\mathbb{R}^n, \qquad P^{(c)} := \sum_{j:\Re a_j = 0} P_j \tag{2.38}
$$

is referred to as the centre *subspace of the fixed point* $x^\star = 0$

The above defined subspaces are invariant subspaces of e^{At}, that is,

$$
\begin{aligned}
e^{\mathsf{A}\,t}\,W_u &\subset W_u \\
e^{\mathsf{A}\,t}\,W_s &\subset W_s \\
e^{\mathsf{A}\,t}\,W_0 &\subset W_0
\end{aligned}
\tag{2.39}
$$

Definition 2.6. *The fixed point is called*

- *a* sink *if* $W_u = W_0 = \{0\}$,
- *a* source *if* $W_s = W_0 = \{0\}$,
- *a* hyperbolic point *if* $W_0 = \{0\}$,
- *an* elliptic point *if* $W_u = W_s = \{0\}$.

3 Linear systems in two dimensions

Let $n = 2$, and let A be in Jordan canonical form, with $\det A \neq 0$. Then we can distinguish between the following behaviours, depending on the eigenvalues a_1, a_2 of A.

1. $a_1 \neq a_2$

   (a) If $a_1, a_2 \in \mathbb{R}$, then

$$\mathsf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \tag{3.1}$$

   and

$$e^{\mathsf{A}\,t} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} y_1(0) \\ e^{a_2 t} y_2(0) \end{bmatrix} \tag{3.2}$$

   The orbits are curves of the form $y_2 = c y_1^{a_2/a_1}$. $x^\star$ is called a *node* if $a_1 a_2 > 0$, and a *saddle* if $a_1 a_2 < 0$.

   (b) If $a_1 = a_2^\dagger = a + \imath\omega \in \mathbb{C}$, then the real canonical form of A is

$$\mathsf{A} = \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix} \tag{3.3}$$

   and

$$e^{\mathsf{A}\,t} = e^{at} \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{at} \,\{ y_1(0)\cos(\omega t) - y_2(0)\sin(\omega t)\} \\ e^{at} \,\{ y_1(0)\sin(\omega t) + y_2(0)\cos(\omega t)\,\} \end{bmatrix} \tag{3.4}$$

   $x^\star$ is called a *focus* if $a \neq 0$, and a *center* if $a = 0$. The orbits are spirals or ellipses.

2. $a_1 = a_2 := a$

   (a) If a has geometric multiplicity 2, then $A = a\,1$ and $e^{\mathsf{A}\,t} = e^{at}\,1$; $x^\star$ is called a *degenerate node*.

   (b) If a has geometric multiplicity 1, then

$$\mathsf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \tag{3.5}$$

   and

$$e^{\mathsf{A}\,t} = e^{at} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{at} \,\{ y_1(0) + y_2(0)t\,\} \\ e^{at} y_2(0) \end{bmatrix} \tag{3.6}$$

   $x^\star$ is called an *improper node*.

3.1 Classification of 2-dimensional linear systems

Consider $\mathsf{A} \in \mathrm{End}(\mathbb{R}^2)$. The most general form is

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.7}$$

The characteristic polynomial is

$$P(\mathsf{A}) = (\lambda - a)(\lambda - d) - bc = \lambda^2 - \lambda(a + d) + a\,d - b\,c \tag{3.8}$$

It can be rewritten in terms of invariants

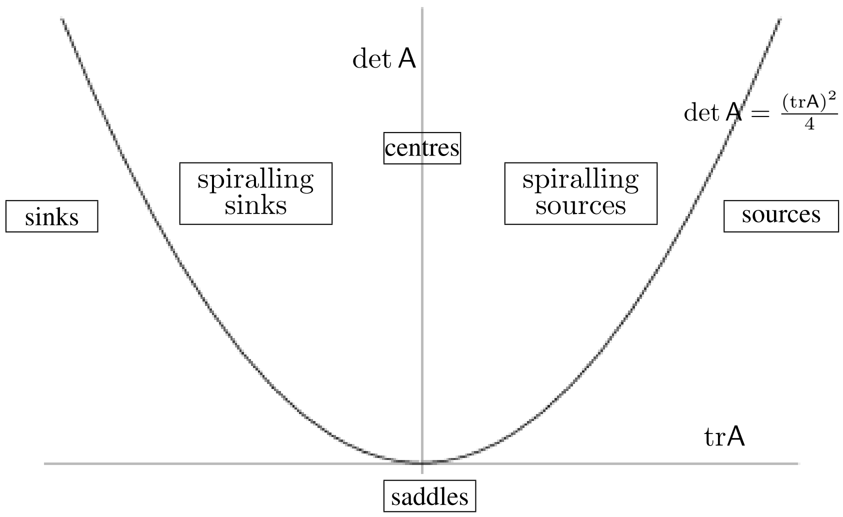
$$P(\mathsf{A}) = \lambda^2 - \lambda\, \mathrm{tr}\mathsf{A} + \det \mathsf{A} \tag{3.9}$$

The eigenvalues are

$$\begin{aligned} \lambda_+ &= \frac{\mathrm{tr}\mathsf{A} + \sqrt{\mathrm{tr}^2\mathsf{A} - 4\det \mathsf{A}}}{2} \\ \lambda_- &= \frac{\mathrm{tr}\mathsf{A} - \sqrt{\mathrm{tr}^2\mathsf{A} - 4\det \mathsf{A}}}{2} \end{aligned} \tag{3.10}$$

one has

- $(\mathrm{tr}\mathsf{A})^2 > 4\,, \det \mathsf{A}$ real eigenvalues
  - $\det \mathsf{A} \ < \ 0$ the origin is a saddle
  - $\det \mathsf{A} \ > \ 0$, $\mathrm{tr}\,\mathsf{A} \ > \ 0$ the origin is a source
  - $\det \mathsf{A} \ > \ 0$, $\mathrm{tr}\,\mathsf{A} \ < \ 0$ the origin is a sink
- $\mathrm{tr}^2\mathsf{A} \ < \ 4\det\ \mathsf{A}$ real eigenvalues
  - $\det \mathsf{A} \ > \ 0$, $\mathrm{tr}\,\mathsf{A} \ > \ 0$ the origin is a spiralling source
  - $\det\ \mathsf{A} \ > \ 0$, $\mathrm{tr}\mathsf{A} \ < \ 0$ the origin is a spiralling sink



(3.11)

A centre is encountered for $\mathrm{tr}\mathsf{A} \ = \ 0$

Appendices

A An extra: 2-d matrices in the Pauli basis

The Pauli matrices

$$\sigma_0 = 1_2\,, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,, \quad \sigma_2 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}\,, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A-1}$$

provide a basis for $\mathrm{End}(\mathbb{C}^2)$. Namely any matrix A in $\mathrm{End}(\mathbb{C}^2)$ can be written as

$$A = \sum_{i=0}^{3} a_i \sigma_i \tag{A-2}$$

Furthermore

$$\begin{aligned} &[\sigma_i\,,\sigma_j] = 2\imath \varepsilon_{ijk} \sigma_k \\ &\{\sigma_i\,,\sigma_j\} := \sigma_i\,\sigma_j + \sigma_j\,\sigma_i = 2\,1_2 \delta_{ij} \qquad \text{anticommutativity} \end{aligned} \tag{A-3}$$

for ε_{ijk} the totally anti-symmetric symbol

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} \tag{A-4}$$

for example

$$\begin{aligned} \sigma_1\sigma_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix} \\ \sigma_2\sigma_1 &= \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix} \end{aligned} \tag{A-5}$$

Proposition A.1. *Using the above algebra*

$$AB = \left(\sum_{i=0}^{3} a_i \sigma_i \right) \left(\sum_{j=0}^{3} b_j \sigma_j \right) = \left(a_0^2 + a \cdot b \right) \sigma_0 + \sum_{i=1}^{3} \left[a_0 b_i + b_0 a_i + \imath \left(a \wedge b \right)_i \right] \sigma_i \tag{A-6}$$

having defined

$$a := \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{A-7}$$

Proof.

$$\left(\sum_{i=0}^{3} a_i \sigma_i \right) \left(\sum_{j=0}^{3} b_j \sigma_j \right) = \sum_{i=1}^{3} a_i b_i \sigma_i^2 + \sum_{i=1}^{3} \sum_{j \neq i} a_i b_j \sigma_i \sigma_j = a \cdot b + \sum_{i=1}^{3} \sum_{j \neq i} a_i b_j \frac{\{\sigma_i\,,\sigma_j\} + [\sigma_i\,,\sigma_j]}{2} \tag{A-8}$$

since

$$\sigma_i \sigma_j = \frac{\{\sigma_i\,,\sigma_j\} + [\sigma_i\,,\sigma_j]}{2} \tag{A-9}$$

Use now the algebra of the Pauli matrices

$$\left(\sum_{i=0}^{3} a_i \sigma_i \right) \left(\sum_{j=0}^{3} b_j \sigma_j \right) = \sum_{i=1}^{3} \sum_{j \neq i} a_i b_j \frac{2\imath \epsilon_{ijk} \sigma_k}{2} = \imath \sum_{i=1}^{3} (a \wedge b)_i \sigma_i \tag{A-10}$$

□

A useful consequence is that

$$A^2 = \left(a_0^2 + a^2\right)\sigma_0 + 2\sum_{i=1}^{3} a_0 a_i \sigma_i$$
$$A^3 = \left(a_0^2 + 3a^2\right) a_0\sigma_0 + \left(3\,a_0^2 + a^2\right)\sum_{i=1}^{3} a_i\sigma_i \tag{A-11}$$

Proposition A.2.

$$A^n = \sum_{k=0}^{\mathrm{int}\left\{\frac{n}{2}\right\}} \binom{n}{2k} a_0^{n-2k} a^{2k}\sigma_0 + \left(\sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}-1} \binom{n}{2k+1} a_0^{n-2k-1} a^{2k}\right)\sum_{i=1}^{3} a_i\sigma_i \tag{A-12}$$

Proof.

$$\begin{aligned} A^{n+1} &= \left[a_0\left(\sum_{k=0}^{\mathrm{int}\left\{\frac{n}{2}\right\}} \binom{n}{2k} a_0^{n-2k} a^{2k}\right) + a^2\left(\sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}-1} \binom{n}{2k+1} a_0^{n-2k-1} a^{2k}\right)\right]\sigma_0 \\ &+ \left[a_0\left(\sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}-1} \binom{n}{2k+1} a_0^{n-2k-1} a^{2k}\right) + \sum_{k=0}^{\mathrm{int}\left\{\frac{n}{2}\right\}} \binom{n}{2k} a_0^{n-2k} a^{2k}\right]\sum_{i=1}^{3} a_i\sigma_i \end{aligned} \tag{A-13}$$

Observe now that

$$\sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}-1} \binom{n}{2k+1} a_0^{n-2k-1} a^{2k+2} = \sum_{k=1}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}} \binom{n}{2k+1} a_0^{n+1-2k} a^{2k} \tag{A-14}$$

and

$$\begin{aligned} &\sum_{k=0}^{\mathrm{int}\left\{\frac{n}{2}\right\}} \binom{n}{2k} a_0^{n+1-2k} a^{2k} + \sum_{k=1}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}} \binom{n}{2k+1} a_0^{n+1-2k} a^{2k} \\ &= a_0^n + \sum_{k=1}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}} \left[\binom{n}{2k} + \binom{n}{2k-1}\right] a_0^{n+1-2k} a^{2k} \\ &= \sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}} \binom{n+1}{2k} a_0^{n+1-2k} a^{2k} \end{aligned} \tag{A-15}$$

analogously for the second term. □

The exponential of $A \in \mathbb{C}\times\mathbb{C}$ can be written as

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{k=0}^{\mathrm{int}\left\{\frac{n}{2}\right\}} \binom{n}{2k} a_0^{n-2k} a^{2k}\sigma_0 + \left(\sum_{k=0}^{\mathrm{int}\left\{\frac{n+1}{2}\right\}-1} \binom{n}{2k+1} a_0^{n-2k-1} a^{2k}\right)\sum_{i=1}^{3} a_i\sigma_i\right] \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left[\sum_{k=0}^{n} \binom{2n}{2k} a_0^{2(n-k)} a^{2k}\sigma_0 + \left(\sum_{k=0}^{n-1} \binom{2n}{2k+1} a_0^{2n-2k-1} a^{2k}\right)\sum_{i=1}^{3} a_i\sigma_i\right] \\ &+ \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left[\sum_{k=0}^{n} \binom{2n+1}{2k} a_0^{2n-2k+1} a^{2k}\sigma_0 + \left(\sum_{k=0}^{n} \binom{2n+1}{2k+1} a_0^{2n-2k} a^{2k}\right)\sum_{i=1}^{3} a_i\sigma_i\right] \end{aligned} \tag{A-16}$$

Observing that

$$\sum_{k=0}^{n}\binom{2n}{2k}a_0^{2(n-k)}a^{2k}=\frac{1}{2}\left[\left(a_0+\sqrt{a^2}\right)^{2n}+\left(a_0-\sqrt{a^2}\right)^{2n}\right]$$
$$\sum_{k=0}^{n}\binom{2n+1}{2k}a_0^{2n-2k+1}a^{2k}=\frac{1}{2}\left[\left(a_0+\sqrt{a^2}\right)^{2n+1}+\left(a_0-\sqrt{a^2}\right)^{2n+1}\right] \tag{A-17}$$

and

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1}a_0^{2n-2k-1}a^{2k}=\frac{1}{2\sqrt{a^2}}\left[\left(a_0+\sqrt{a^2}\right)^{2n}-\left(a_0-\sqrt{a^2}\right)^{2n}\right]$$
$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}a_0^{2n-2k}a^{2k}=\frac{1}{2\sqrt{a^2}}\left[\left(a_0+\sqrt{a^2}\right)^{2n+1}-\left(a_0-\sqrt{a^2}\right)^{2n+1}\right] \tag{A-18}$$

Which finally yields

$$e^A=\frac{e^{a_0+\sqrt{a^2}}+e^{a_0-\sqrt{a^2}}}{2}\sigma_0+\frac{e^{a_0+\sqrt{a^2}}-e^{a_0-\sqrt{a^2}}}{2\sqrt{a^2}}\sum_{i=1}^{3}a_i\sigma_i \tag{A-19}$$

In particular if

$$\mathrm{tr}A=0 \qquad \Rightarrow \qquad a_0=0 \tag{A-20}$$

then

$$e^A=\cosh\left(\sqrt{a^2}\right)\sigma_0+\frac{\sinh\left(\sqrt{a^2}\right)}{\sqrt{a^2}}\sum_{i=1}^{3}a_i\sigma_i \tag{A-21}$$

References

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