Lecture 01: Existence and uniqueness theorems and flows

Introduction

The expounded material can be found in

- ch. 2 of N. Berglund's lecture notes [1]
- ch. 1 of [2]

1 Existence and uniqueness theorems

Let $D \subset \mathbb{R}^n$. For any $\boldsymbol{x} \in D$ we consider the ordinary differential equation

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t) \tag{1.1a}$$

$$\boldsymbol{x}_0 = \boldsymbol{x} \tag{1.1b}$$

driven by the vector field

$$\boldsymbol{f} \colon D \mapsto \mathbb{R}^n \tag{1.2}$$

We denote by $\mathrm{C}^r(D;\mathbb{R}^n) \equiv \mathrm{C}^r(D)$ the space of $\mathbb{R}^n$-valued vector fields with support over D which are r-times continuously differentiable. We then posit $\boldsymbol{f} \in \mathrm{C}^r(D)$ for some $r \geq 0$.

A differential problem (1.1) specified by an *evolution law* (1.1a) and an initial data (1.1b) is also often called a *Cauchy problem*. Equation (1.1b) fixes the *initial data* or *initial condition* for $t = 0$.

1.1 Existence

Theorem 1.1 (**Peano–Cauchy**). *Let $\boldsymbol{f} \in \mathrm{C}(D)$ i.e. let $\boldsymbol{f}$ be a continuous vector field over D. For every $\boldsymbol{x} \in D$, there exists at least one local solution of (1.1) through $\boldsymbol{x}$, that is, there is an open interval I such that $0 \in I$ and a function*

$$\boldsymbol{x}_t \colon I \mapsto D \tag{1.3}$$

such that (1.1a) and (1.1b) are satisfied for $t \in I$.

The support I of a solution $\boldsymbol{x}_t$ can be extended as long as $\boldsymbol{x}_t$ remains finite.

Proposition 1.1. *Every solution*

$$\boldsymbol{x}_t \colon I \mapsto D \tag{1.4}$$

starting from $\boldsymbol{x}_0 = \boldsymbol{x}$ can be continued to a maximal interval of existence $I_{\max} = (t_{\min}, t_{\max})$ comprising the time $t = 0$ when the initial condition is assigned. If $t_{\max} < \infty$ or $t_{\min} > -\infty$, then for any compact $K \subset D$ exists a time $t \in I_{\max}$ such that $\boldsymbol{x}_t \notin K$.

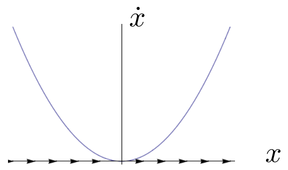
In other words, the proposition states that either there is a time t for which

$$\boldsymbol{x}_t \in \partial D \tag{1.5}$$

i.e. reaches the boundary of D or the solution diverges. Very simple vector fields (1.2) may bring about explosions.

Example 1.1. For example

$$\dot{x}_t = \alpha\, x_t^2 \tag{1.6}$$



admits the solution

$$x_t = \frac{1}{\frac{1}{x} - \alpha\, t} \tag{1.7}$$

For any $x \in \mathbb{R}_+$ the solution diverges in finite time:

$$t_{\text{expl}} = \frac{1}{\alpha\, x} \tag{1.8}$$

Note the physical dimension of the problem

$$\begin{aligned} [x] &= [space] \\ [t] &= [time] \end{aligned} \quad \Rightarrow \quad [\alpha] = \left[\frac{1}{time \times space}\right] = -[space] - [time] \tag{1.9}$$

The vector field (1.2) is ("time")-autonomous. This means that it does not depend explicitly upon the "time" variable with respect to which we differentiate $\boldsymbol{x}_t$.

Proposition 1.2. *Let*

$$\boldsymbol{x}_t = \boldsymbol{\phi}(t, \boldsymbol{x}) \equiv \boldsymbol{\phi}_t(\boldsymbol{x}) \tag{1.10}$$

be the unique solution of (1.1). Then $\tilde{\boldsymbol{x}}_t = \boldsymbol{x}_{t-t_o}$ is the unique solution of the Cauchy problem evolving according to (1.1a) from the initial data

$$\tilde{\boldsymbol{x}}_{t_o} = \boldsymbol{x} \tag{1.11}$$

assigned at $t_o \neq 0$.

Proof. By (1.10)

$$\tilde{\boldsymbol{x}}_t = \boldsymbol{\phi}(t - t_o, \boldsymbol{x}) \equiv \boldsymbol{\phi}_{t-t_o}(\boldsymbol{x}) \tag{1.12}$$

hence for any $t_o \in \mathbb{R}$

$$\dot{\boldsymbol{x}}_{t-t_o} = \frac{d\boldsymbol{\phi}}{dt}(t - t_o, \boldsymbol{x}) = \frac{d\boldsymbol{\phi}}{du}(u, \boldsymbol{x})\bigg|_{u=t-t_o} = \boldsymbol{f}(\boldsymbol{x}_u)|_{u=t-t_o} = \boldsymbol{f}(\boldsymbol{x}_{t-t_o}) \tag{1.13}$$

which yields the claim. □

The important consequence of the above proposition is that solutions of autonomous differential equations depend only upon the time *elapsed* from the moment t_{o} the initial condition is assigned but not upon t_{o} itself. Non-autonomous systems such as

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, t) \tag{1.14}$$

can always be regarded as autonomous systems in a space with one dimension more. Upon defining

$$u = t \qquad \Rightarrow \qquad \dot{f}(t) = \frac{df}{dt}(t) \equiv \frac{df}{du}(u) \tag{1.15}$$

we can write

$$\dot{\boldsymbol{x}}_u = \boldsymbol{f}(\boldsymbol{x}_t, t) \tag{1.16a}$$

$$\dot{t}_u = 1 \tag{1.16b}$$

Hence if we define $E = D \times I \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ with

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{x} \\ t \end{bmatrix} \in E \qquad \& \qquad \boldsymbol{g} = \begin{bmatrix} \boldsymbol{f} \\ 1 \end{bmatrix} : E \mapsto \mathbb{R}^{n+1} \tag{1.17}$$

we arrive to the *autonomous* dynamical system

$$\dot{\boldsymbol{y}}_u = \boldsymbol{g}(\boldsymbol{y}_u) \tag{1.18a}$$

$$\boldsymbol{y}_0 = \begin{bmatrix} \boldsymbol{x} \\ 0 \end{bmatrix} \tag{1.18b}$$

1.2 Uniqueness

Theorem 1.2 (Picard–Lindelöf). *Suppose that $\boldsymbol{f} \in \mathrm{C}(D)$ and locally Lipschitzian that is, for every compact $K \subset D$ there exists a constant $L_K \in \mathbb{R}_+$ such that*

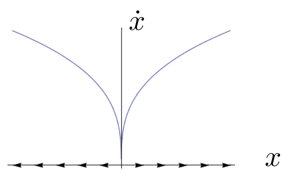
$$\| \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y}) \| \leq L_K \| \boldsymbol{x} - \boldsymbol{y} \| \tag{1.19}$$

for all $\boldsymbol{x}$ and $\boldsymbol{y}$ belonging to K. Then there is a unique solution of (1.1) for each initial data in D

We notice that a differentiable function satisfies the Lipschitz condition.

Example 1.2. The ordinary differential equation

$$\dot{x}_t = \alpha \, |x_t|^{1/3} \tag{1.20}$$



is driven by a vector field non differentiable in zero. In general we can integrate (1.20) and obtain

$$x_t = \left(x^{2/3} + \frac{2}{3} \alpha \, t \right)^{3/2} \tag{1.21}$$

If, however, we assign the initial data to be $x = 0$ we can find the solutions (1.21), and also

$$x_t = 0 \tag{1.22}$$

We can combine these solutions together and generate a one parameter t_* family of solutions taking the form

$$x_t = \begin{cases} 0 & 0 \leq t \leq t_* \\ \left(x^{2/3} + \frac{2}{3} \alpha \, t \right)^{3/2} & t_* < t \end{cases} \tag{1.23}$$

An important implication of the Picard–Lindelöf theorem is that *for any finite time* different solutions of (1.1) driven by a smooth vector field cannot intersect. Namely we have

Proposition 1.3. *For any $\boldsymbol{x} \in D$ there exists only one solution of (1.1) with $\boldsymbol{f} \in C^r(D)$, $r \geq 1$ passing through this point.*

Proof. Let $\boldsymbol{x}_t^{(i)}$, $i = 1, 2$ solutions of (1.1) satisfying

$$\begin{aligned} \boldsymbol{x}_{t_1}^{(1)} &= \boldsymbol{x} \\ \boldsymbol{x}_{t_2}^{(2)} &= \boldsymbol{x} \end{aligned} \tag{1.24}$$

Since (1.1) is autonomous

$$\tilde{\boldsymbol{x}}_t = \boldsymbol{x}_{t-(t_1-t_2)} \tag{1.25}$$

is also a solution of (1.1) satisfying

$$\tilde{\boldsymbol{x}}_{t_1} = \boldsymbol{x}_{t_2}^{(2)} = \boldsymbol{x} = \boldsymbol{x}_{t_1}^{(1)} \tag{1.26}$$

We thus see that our starting assumption leads to a contradiction with the Picard–Lindelöf theorem. □

1.3 Smooth dependence upon initial data

Let us now suppose that the vector field $\boldsymbol{f}$ in (1.1a) carries a *parametric* dependence upon a set of parameters. Specifically let us suppose that

$$\boldsymbol{f} \in \mathrm{C}^r(D_1 \times D_2; \mathbb{R}^n) \tag{1.27}$$

We will denote by $\boldsymbol{x}$ coordinates over $D_1 \subset \mathbb{R}^n$ and by $\boldsymbol{y}$ coordinates over $D_2 \subset \mathbb{R}^m$ so that

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \tag{1.28}$$

Note that we are not assuming any relation between the integers m and n.

Theorem 1.3 (Picard–Lindelöf). *Suppose that $r \geq 1$ in (1.27). In words: $\boldsymbol{f}$ is at least once continuously differentiable with respect to all its arguments. Let I the maximal interval of existence of the unique solution of the ordinary differential equation*

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t; \boldsymbol{y}) \tag{1.29a}$$

$$\boldsymbol{x}_0 = \boldsymbol{x} \tag{1.29b}$$

Under these hypotheses we can write the unique solution of (1.29) as

$$\boldsymbol{x}_t = \boldsymbol{\phi}(t, \boldsymbol{x}; \boldsymbol{y}) \equiv \boldsymbol{\phi}_t(\boldsymbol{x}; \boldsymbol{y}) \tag{1.30}$$

where

$$\boldsymbol{\phi}_t \in C^r(I \times D_1 \times D_2\,, D_1) \tag{1.31}$$

Thus, under smoothness assumptions over the vector field $\boldsymbol{f}$ we are entitled to consider derivatives of solutions (1.29) with respect to *all* its parametric dependencies. Moreover, if we generically denote by $\boldsymbol{x}'_t$ the derivative of $\boldsymbol{x}_t$ with respect to any of its parameters the same notation may apply to three different cases:

1. $\boldsymbol{\xi}'_t \in \mathbb{R}^{n\times m}$ if

$$\boldsymbol{\xi}'_t = \partial_{\boldsymbol{y}} \otimes \boldsymbol{\xi}_t \equiv \partial_{\boldsymbol{y}} \otimes \boldsymbol{\Phi}_t(\boldsymbol{x};\boldsymbol{y}) \tag{1.32}$$

2. $\boldsymbol{\xi}'_t \in \mathbb{R}^{n^2}$ if

$$\boldsymbol{\xi}'_t = \partial_{\boldsymbol{x}} \otimes \boldsymbol{\xi}_t \equiv \partial_{\boldsymbol{x}} \otimes \boldsymbol{\Phi}_t(\boldsymbol{x};\boldsymbol{y}) \tag{1.33}$$

3. $\boldsymbol{\xi}'_t \in \mathbb{R}^{n}$ if

$$\boldsymbol{\xi}'_t = \dot{\boldsymbol{\xi}}_t \equiv \dot{\boldsymbol{\Phi}}_t(\boldsymbol{x};\boldsymbol{y}) \tag{1.34}$$

For the first case we have that

$$\dot{\boldsymbol{\xi}}'_t = \boldsymbol{\xi}'_t \cdot \partial_{\boldsymbol{\xi}_t} \boldsymbol{f}(\boldsymbol{\xi}_t;\boldsymbol{y}) + \partial_{\boldsymbol{y}} \otimes \boldsymbol{f}(\boldsymbol{\xi}_t,\boldsymbol{y})$$

as the dependence upon the parameters $\boldsymbol{y}$ is explicit. In the remaining two cases, the dependence upon the parameters is implicit through the trajectory. We find that

$$\dot{\boldsymbol{\xi}}'_t = \boldsymbol{\xi}'_t \cdot \partial_{\boldsymbol{\xi}_t} \boldsymbol{f}(\boldsymbol{\xi}_t) \tag{1.35}$$

For all the above cases it is expedient to define

$$\mathsf{A}_t^{ij} = (\partial_{\boldsymbol{x}_t^j} \otimes \boldsymbol{f}^i)(\boldsymbol{x}_t) \tag{1.36}$$

In other words, A defines an endomorphism of $\mathbb{R}^n$: with slight abuse of language $\mathsf{A} \in \mathrm{End}(\mathbb{R}^n)$.

Proposition 1.4. *Let* t *denote the transposition operation, then*

$$\mathsf{A}_t := (\partial_{\boldsymbol{x}_t} \otimes \boldsymbol{f})(\boldsymbol{x}_t) \tag{1.37}$$

and

$$\mathsf{F}_t := (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \tag{1.38}$$

then

$$\dot{\mathsf{F}}_t = \mathsf{A}_t\, \mathsf{F}_t \tag{1.39}$$

Proof. By hypothesis

$$\frac{\mathrm{d}\boldsymbol{\phi}_t}{\mathrm{d}t}(\boldsymbol{x}) = \lim_{\mathrm{d}t\downarrow 0} \frac{\boldsymbol{\phi}_{t+\mathrm{d}t}(\boldsymbol{x}) - \boldsymbol{\phi}_t(\boldsymbol{x})}{\mathrm{d}t} = \lim_{\mathrm{d}t\downarrow 0} \frac{\boldsymbol{\phi}_{\mathrm{d}t} \circ \boldsymbol{\phi}_t(\boldsymbol{x}) - \boldsymbol{\phi}_t(\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{f} \circ \boldsymbol{\phi}_t(\boldsymbol{x}) \tag{1.40}$$

hence it follows

$$\begin{aligned} \partial_{\boldsymbol{x}} \otimes \dot{\boldsymbol{\phi}}_t(\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}t} \partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t(\boldsymbol{x}) = \\ (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \cdot \partial_{\boldsymbol{\phi}_t(\boldsymbol{x})} \boldsymbol{f} \circ \boldsymbol{\phi}_t(\boldsymbol{x}) = (\partial_{\boldsymbol{\phi}_t(\boldsymbol{x})} \otimes \boldsymbol{f})^{\mathsf{t}} \circ \boldsymbol{\phi}_t(\boldsymbol{x}) \cdot (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi})_t(\boldsymbol{x}) = \mathsf{A}_t\, \mathsf{F}_t \end{aligned} \tag{1.41}$$

□

Of particular interest is the variation with respect to initial data

Definition 1.1. *We will refer to the quantity*

$$\mathsf{F}_t = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \tag{1.42}$$

as the fundamental solution of the linearized dynamics.

By construction (1.42) satisfies the initial condition

$$\mathsf{F}_0 = 1 \tag{1.43}$$

Namely from

$$\boldsymbol{\phi}_0(\boldsymbol{x}) = \boldsymbol{x} \tag{1.44}$$

it follows that

$$\mathsf{F}_0 \equiv \partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_0(\boldsymbol{x}) = 1 \tag{1.45}$$

Example 1.3. Let us consider again

$$\dot{x}_t = \alpha\, x_t^2 \qquad \Rightarrow \qquad x_t = \frac{1}{\frac{1}{x} - \alpha\, t} \tag{1.46}$$

We have then

$$\partial_x x_t = \frac{1}{(\frac{1}{x} - \alpha\, t)^2\, x^2} := x'_t \tag{1.47}$$

which satisfies

$$\dot{x}'_t = \frac{2\,\alpha}{(\frac{1}{x} - \alpha\, t)^3\, x^2} = 2\,\alpha\, x_t\, x'_t \tag{1.48}$$

as expected form direct differentiation of the ordinary differential equation in (1.46).

2 Flows: terminology

Definition 2.1 (Homeomorphism)**.** *Let $D \subset \mathbb{R}^n$ be an open domain. A map $\boldsymbol{\phi}\colon\ D \mapsto D$ is called an* homeomorphism *if it is continuous and admits a continuous inverse.*

Definition 2.2 (Diffeomorphism)**.** *Let $D \subset \mathbb{R}^n$ be an open domain. A map $\boldsymbol{\phi}\colon\ D \mapsto D$ is called a* diffeomorphism *if it is continuously* differentiable *and admits a continuously* differentiable *inverse.*

Definition 2.3. *Similarly, for all $r \geq 1$, a C^r*-diffeomorphism *is an invertible map $\boldsymbol{\phi}\colon\ D \mapsto D$ such that both $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^{-1}$ admit continuous derivatives up to order r.*

Definition 2.4 (flow)**.** *A one parameter family of diffeomorphisms Φ_t*

$$\Phi : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n \tag{2.1}$$

such that

$$\begin{aligned} &\Phi_0 = 1 \\ &\Phi_t \Phi_s = \Phi_{t+s} \end{aligned} \tag{2.2}$$

is called flow.

The definition of flow implies $\Phi_{-t} = \Phi_t^{-1}$ and therefore that Φ forms a one parameter t group of transformations. The combined use of the existence and uniqueness theorems for (1.1) allows us to conclude that

Proposition 2.1. *The solution of (1.1) regarded as the map $\boldsymbol{\phi}\colon \mathbb{R} \times D \mapsto D$ is a flow to which we will also refer as the fundamental solution of (1.1).*

3 Evolution of volumes

Proposition 3.1. *Let $\boldsymbol{f} \in \mathrm{C}^r(D)$ with $r \geq 1$ and let $M \subset D$. Let us denote by M_t the image of M through the flow $\boldsymbol{\phi}$ of (1.1) with $\boldsymbol{f}$ sufficiently regular. Let us define*

$$V_t^{[M]} = \mathrm{Vol}(M_t) \tag{3.1}$$

then we have

$$\dot{V}_t^{[M]} = \int_{M_t} \mathrm{d}^n x \, (\partial_{\boldsymbol{x}} \cdot \boldsymbol{f})(\boldsymbol{x}) \tag{3.2}$$

Proof. The flow being a diffeomorphism means that

1. $\boldsymbol{\phi}$ is differentiable with the initial data

$$V_t^{[M]} = \int_{M_t} \mathrm{d}^n x = \int_M \mathrm{d}^n x \; |\det(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x})| \tag{3.3}$$

2. $\boldsymbol{\phi}$ is invertible thus

$$\boldsymbol{\phi}_0(\boldsymbol{x}) = \boldsymbol{x} \tag{3.4}$$

implies

$$\det(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_0)(\boldsymbol{x}) = \det \mathsf{1} = 1 \tag{3.5}$$

and

$$\det(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \; > \; 0 \tag{3.6}$$

The strict inequality holds because otherwise

$$\mathsf{F}_t = (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \tag{3.7}$$

solving the associated linearized dynamics

$$\dot{\boldsymbol{x}}' = \boldsymbol{x}' \cdot \partial_{\boldsymbol{x}_t} \boldsymbol{f}(\boldsymbol{x}_t) \equiv (\partial_{\boldsymbol{x}_t} \otimes \boldsymbol{f})(\boldsymbol{x}_t) \cdot \boldsymbol{x}' \tag{3.8}$$

would not be invertible. This is a contradiction because F_t is the unique fundamental solution of the linearized dynamics.

We arrived at

$$V_t^{[M]} = \int_{M_t} \mathrm{d}^n x = \int_M \mathrm{d}^n x \; \det(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)(\boldsymbol{x}) \tag{3.9}$$

We observe that for any strictly positive matrix A the equality

$$\ln \det \mathsf{A} = \operatorname{tr} \ln \mathsf{A} \tag{3.10}$$

holds true. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr} \ln \partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t &= \operatorname{tr}(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t) \\ &= \operatorname{tr}(\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t)^{-1} (\partial_{\boldsymbol{x}} \otimes \boldsymbol{\phi}_t) \, \partial_{\boldsymbol{x}_t} \otimes \boldsymbol{f}(\boldsymbol{x}_t) = \operatorname{tr} \, \partial_{\boldsymbol{x}_t} \otimes \boldsymbol{f}(\boldsymbol{x}_t) = (\partial_{\boldsymbol{x}_t} \cdot \boldsymbol{f})(\boldsymbol{x}_t) \end{aligned} \tag{3.11}$$

□

References

[1] N. Berglund. Geometrical theory of dynamical systems. ETH lecture notes, 2001, arXiv:math/0111177.

[2] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2 of *Texts in applied mathematics*. Springer, 2 edition, 2003.