Remarks on Problem 8.2

At the exercise session, there was some confusion regarding the use of Sommerfeld's radiation condition in the proof of

$$\int\limits_{\partial B_R} \left(u\,\frac{\partial}{\partial r}\Phi_k(x,\cdot) - \frac{\partial u}{\partial r}\,\Phi_k(x,\cdot)\right) \xrightarrow[R\longrightarrow\infty]{} 0.$$

Instead of using Sommerfeld's finiteness condition, which says that $u(x) \ll \frac{1}{|x|}$ as $|x| \longrightarrow \infty$, and which follows from the problem under consideration, one can use Sommerfeld's radiation condition directly in the following form (below we follow the excellent lecture notes of R. Kress on Inverse Scattering Theory):

$$\int\limits_{\partial B_R} \left|\frac{\partial u}{\partial r} - iku\right|^2 \ll R^2 \max_{\partial B_R} \left|\frac{\partial u}{\partial r} - iku\right|^2 \xrightarrow[R\longrightarrow\infty]{} 0.$$

Noting that

$$2k\Im \int\limits_{\partial B_R} \overline{\frac{\partial u}{\partial r}}\, u = 2k\Im \int\limits_{B_R\setminus\overline{\Omega}} \overline{\Delta u}\, u + 2k\Im \int\limits_{\partial\Omega} \overline{\frac{\partial u}{\partial \nu}}\, u,$$

and that here the first term must vanish since the volume integral is real-valued, we deduce that

$$\int\limits_{\partial B_R} \left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2\, |u|^2 \right) \xrightarrow[R\longrightarrow\infty]{} -2k\Im \int\limits_{\partial\Omega} \overline{\frac{\partial u}{\partial \nu}}\, u \ll 1.$$

This shows that

$$\int\limits_{\partial B_R} |u|^2 \ll 1,$$

which can be combined with the Cauchy–Schwarz inequality to get

$$\begin{aligned}
&\int\limits_{\partial B_R} \left(u\,\frac{\partial}{\partial r}\Phi_k(x,\cdot) - \frac{\partial u}{\partial r}\,\Phi_k(x,\cdot)\right) \\
&\quad = \int\limits_{\partial B_R} u \left(\frac{\partial}{\partial r}\Phi_k(x,\cdot) - ik\Phi_k(x,\cdot)\right) - \int\limits_{\partial B_R} \left(\frac{\partial u}{\partial r} - iku \right) \Phi_k(x,\cdot) \\
&\quad \ll \sqrt{\int\limits_{\partial B_R} |u|^2} \sqrt{\int\limits_{\partial B_R} \left| \frac{\partial}{\partial r}\Phi_k(x,\cdot) - ik\Phi_k(x,\cdot)\right|^2} \\
&\qquad\qquad + \sqrt{\int\limits_{\partial B_R} \left|\frac{\partial u}{\partial r} - iku\right|^2} \sqrt{\int\limits_{\partial B_R} |\Phi_k(x,\cdot)|^2} \\
&\quad \ll 1\cdot o(1) + o(1)\cdot 1 = o(1),
\end{aligned}$$

where $R \longrightarrow \infty$, which is what was to be proved.