Kirjallisuus

Integraaliyhtälöistä:

- Colton - Kress: Integral Equation Methods in Scattering Theory, Wiley
- Hochstadt: Integral Equations, Wiley
- Kress: Linear Integral Eqns, Springer
- Yosida: Lectures on differential and Integral Eqns, Dover Edition available

Tavallisista diff.yhtälöistä:

- Colton: Partial Diff. Eqns, Wiley
- Evans: - " - , AMS

Funktionaalianalyysia:
& kompleksianalyysia

- Dieudonne: Foundations of Modern Analysis, Acad. Press
- Rudin: Functional Analysis ← McGraw Hill
- " : Real and Complex Analysis ↙
- Yosida: Functional Analysis: Springer

INTEGRAALIYHTÄLÖT

S. 2012 / HY

P. Ola

0. Johdanto

Integraaliyhtälöt ja niiden tutkimus ovat näytelleet keskeistä osaa matematiikan sovelluksissa monissa fysiikan, biologian ja insinööritieteiden ongelmissa. Edelleen ne ovat merkittävällä tavalla vaikuttaneet modernin funktionaalianalyysin syntyyn.

Esim. 0.1. Jos ρ on varausjakauma $\mathbb{R}^3$:ssa, niin (staattinen) sähköinen potentiaali u toteuttaa Poissonin yhtälön

(0.1.1) $\Delta u = -\rho/\varepsilon_0$, ε_0 = tyhjiön permittiivisyys

Oletetaan lisäksi että $\rho \equiv 0$ raj.-joukon komplementissa ja että $u(x) \to 0$ kun $|x| \to \infty$ vähintään kuten $1/|x|$. Tällöin (0.1.1) ~~on ekvivalentti~~ saadaan ratkaistua

$$u(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi\varepsilon_0 |x-y|}\,dy \quad \text{(Poissonin kaava)}$$

Tarkastellaan samaa ongelmaa mutta $\mathbb{R}^3$:n raj. alueessa D; vaadimme lisäksi että reuna ∂D on maadoitettu eli:

$$\begin{cases} u|_{\partial D} = 0. \\ \Delta u = -\rho/\varepsilon_0 \quad D\text{:ssä}, \quad \operatorname{supp}\rho \subset D \end{cases}$$

Tälle löytyy ratkaisukaava suljetussa muodossa vain kun D on sopivasti symmetrinen kappale esim pallo tai kuutio. Mitä voimme tehdä yleisemmässä tilanteessa?

Olkoon u_0 vastaava koko avaruuden ratkaisu:

$$u_0(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi\varepsilon_0|x-y|}\,dy$$

ja etsitään u:ta muodossa

$$u = u_0 + v.$$

Nyt

$$-\rho/\varepsilon_0 = \Delta u = \underbrace{\Delta u_0}_{=-\rho/\varepsilon_0} + \Delta v \quad \Rightarrow \quad \underline{\Delta v = 0}$$

$$0 = u|_{\partial D} = u_0|_{\partial D} + v|_{\partial D} \quad \Rightarrow \quad v|_{\partial D} = \underbrace{-u_0|_{\partial D}}_{\text{tunnettu.}} =: f$$

Saamme siis ongelman

$$\begin{cases} \Delta v = 0 \quad D\text{:ssä} \\ v|_{\partial D} = f \end{cases} \qquad \text{"Dirichlet-ongelma"}$$

Tämä voidaan ratkaista käyttäen integraaliyhtälöitä, useammallakin eri tavalla:

Kirjoitetaan v muodossa

$$v(x) = \int_{\partial D} \frac{\varphi(y)}{4\pi |x-y|} \, dS(y)$$

"ns. yksikerrospotentiaali", φ varausjakauma ∂D:llä ↑ tuntematon

Tällöin φ:lle saadaan ns. Fredholmin 1. lajin integraaliyhtälö

(FI) $$\int_{\partial D} \frac{1}{|x-y|} \varphi(y) \, dS(y) = 4\pi f(x) \quad \forall x \in \partial D$$

Voimme myös tehdä v:lle avauksen (tuntemattoman) dipoli-jakauman $\varphi(y)$, $y \in \partial D$, avulla muodossa

$$v(x) = \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \, dS(y), \quad \nu = \partial D\text{:n ulk. ulkonormaali}$$

Tämä johtaa integraaliyhtälöön "kaksikerrospotentiaali"

(FII) $$\frac{1}{2}\varphi(x) + \frac{1}{4\pi} \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \, dS(y) = f.$$

Tämä on ns. 2. lajin Fredholm integraaliyhtälö.

Molemmilla on omat etunsa, ja jos molemmat ovat yksikäsitteisesti ratkeavia, niin v määräytyy yksikäsitteisesti. Näin ei kuitenkaan välttämättä ole, joten haluamme ymmärtää FI:n ja FII:n rakennetta paremmin.

Esim. 0.2 Tarkastellaan tavallista differentiaaliyhtälöä:

$$y' = f(x,y).$$

Integroimalla tämä puolittain yli välin $[x_0, x]$ saamme

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s,y) \, ds.$$

Huomaa että tämä on yleensä epälineaarinen tuntemattoman y suhteen. Oletetaan että

$$f(s, y(s)) = k(s) y(s) \quad \text{jollain sopivalla } k.$$

Tällöin saamme lineaarisen int. yhtälön

(VII) $$y(x) = \overset{y_0}{y(x_0)} + \int_{x_0}^{x} k(s) y(s) \, ds.$$

Tämä on yksinkertainen esimerkki ns. toisen lajin Volterra-yhtälöstä, ja se vastaa siis diff. yhtälön alkuarvo-ongelmaa

$$\begin{cases} y' = k(x) y \\ y(x_0) = y_0 \end{cases}$$

(Tämä on helppo ratkaista separoimalla muuttujat – helpompi kuin (VII):n)

Yleinen 2. lajin Volterra yhtälö on muotoa

$$y(x) = f(x) + \int_{x_0}^{x} k(s,x) y(s) \, ds$$

sopivalla k.

Kun k & f riitt. sään. vastaa tämä ns. integro-diff. yht.

$$y' = f' + k(x,x) y + \int_{x_0}^{x} \frac{\partial k}{\partial x}(s,x) y(s) \, ds$$

I VOLTERRA YHTÄLÖT

Volterran yhtälöt on nimetty Italialaisen matemaatikon Vito Volterran (1860-1940) mukaan (*. Hän tutki näitä yhtälöitä 1900-luvun ensi vuosina.

1.1. 2. Lajin yhtälöiden yleiskäsitteinen ratkaisuns

Olk. $K \in C([a,b]\times[a,b])$, $f \in C([a,b])$ (molemmat voivat olla kompleksiarvoisia). Haluamme löytää jatkuvan funktion $\varphi \in C([a,b])$ s.e.

(1.1.1) $$\varphi(s) - \lambda\int_a^s K(s,t)\varphi(t)\,dt = f(s);$$

tässä $\lambda \in \mathbb{C}$ on parametri. Yhtälöä (1.1.1) kutsutaan 2. lajin Volterra-yhtälöksi funktiolle φ.

Tehdään aluksi formaali päättely: haetaan φ:tä sarjana

(1.1.2) $$\varphi(s) = \sum_{n=0}^{\infty}\lambda^n\varphi_n(s)$$ joillain funktioille φ_n.

Oletetaan että (1.1.2) suppenee itseisesti kun $|\lambda| \le \lambda_0$ välillä $[a,b]$ joillain $\lambda_0 > 0$.

*) 1931 Volterra oli yksi 12:sta yliopisto proff:sta jotka kieltäytyivät vannomasta uskollisuusvalaa Benito Mussolinille; muut 1238 vannoivat.

Sijoitetaan (1.1.2) (1.1.1):een:

$$\sum_{n=0}^{\infty}\lambda^n\varphi_n(s) - \lambda\int_a^s K(s,t)\sum_{n=0}^{\infty}\lambda^n\varphi_n(t)\,dt = f(s)$$

$$\Leftrightarrow \varphi_0(s) + \lambda\Big(\varphi_1(s) - \int_a^s K(s,t)\varphi_0(t)\,dt\Big) + \lambda^2\Big(\varphi_2(s) - \int_a^s K(s,t)\varphi_1(t)\,dt\Big)$$
$$+ \dots + \lambda^n\Big(\varphi_n(s) - \int_a^s K(s,t)\varphi_{n-1}(t)\,dt\Big) + \dots = f(s).$$

Tämä toteutuu $\forall\,\lambda$, $|\lambda| \le \lambda_0$ joss

(1.1.3) $$\begin{cases} \varphi_0(s) = f(s) \\ \varphi_1(s) = \int_a^s K(s,t)\varphi_0(t)\,dt \\ \vdots \\ \varphi_n(s) = \int_a^s K(s,t)\varphi_{n-1}(t)\,dt \\ \vdots \end{cases}$$

Kaavat (1.1.3) antavat siis iteratiivisen reseptin kuinka löytää φ_n:t kun f ja K on annettu. Osoitamme, että (1.1.3) ja (1.1.2) itse asiassa määräävät (1.1.1):n ratkaisun $\forall\,\lambda \in \mathbb{C}$:

Lause 1.1.1. Kun $f \in C([a,b])$, $K \in C([a,b]\times[a,b])$ ja φ_n:t on määritelty (1.1.3):lla, niin $\varphi(s) = \sum_{n=0}^{\infty}\lambda^n\varphi_n(s)$ on itseisesti suppeneva $C([a,b])$:ssä $\forall\,\lambda \in \mathbb{C}$ ja määrää yhtälön (1.1.1) ratkaisun.

Tod. Olk. $M = \sup_{[a,b]}|f(s)|$, $N = \sup_{[a,b]\times[a,b]}|K(s,t)|$

04.09.
12

Tällöin

$$|\varphi_0(s)| = |f(s)| \le M, \quad a \le s \le b$$

$$|\varphi_1(s)| \le \int_a^s |K(s,t)||\varphi_0(t)|\,dt \le MN(s-a)$$

$$|\varphi_2(s)| \le \int_a^s |K(s,t)||\varphi_1(t)|\,dt \le N\int_a^s MN(t-a)\,dt$$

$$= MN^2(s-a)^2/2 .$$

Induktiolla n:n suhteen saamme ey:n

$$|\varphi_n(s)| \le MN^n(s-a)^n/n!$$

⌈Tod: Pätee kun $n=0$ (& $n=1,2$). Ol. että

$$|\varphi_{n-1}(s)| \le MN^{n-1}(s-a)^{n-1}/(n-1)! .$$

Tällöin

$$|\varphi_n(s)| \le \int_a^s |K(s,t)||\varphi_{n-1}(t)|\,dt \le \frac{MN^n}{(n-1)!}\int_a^s (t-a)^{n-1}\,ds = \frac{MN^n}{n!}(s-a)^n .$$ ⌋

Siis

$$\lambda^n|\varphi_n(s)| \le M(\lambda N(s-a))^n/n! ,$$

joten vert. periaate $\left(\sum_{n=0}^{\infty} \frac{\lambda N(s-a)^n}{n!} = e^{\lambda N(s-a)} \right)$

$\Rightarrow$ sarja $\sum_0^\infty \lambda^n \varphi_n(s)$ suppenee itse & tas. välillä $[a,b]$ ja

$\varphi(s) = \sum_0^\infty \lambda^n \varphi_n(s)$ on jv. $[a,b]$:ssä.

Se että φ on (1.1.1):n ratkaisu seuraa suoraan yht. (1.1.3) johtamisesta. □

Tämä todistaa siis olemassaolon (ja antaa myös algoritmin ratkaisun numeeriseksi laskemiseksi sarjakehitelmän avulla).

Todistetaan seuraavaksi yksikäsitteisyys:

<u>Lause 1.1.2.</u> yht. (1.1.1) on korkeintaan yksi ratkaisu $\varphi \in C([a,b])$.

Tod. Ol. että $\varphi_1, \varphi_2 \in C([a,b])$ toteuttavat (1.1.1):n

Olk. $\psi = \varphi_1 - \varphi_2$. Tällöin

$$\varphi_1(s) - \lambda\int_a^s K(s,t)\varphi_1(t)\,dt = f(s) = \varphi_2(s) - \lambda\int_a^s K(s,t)\varphi_2(t)\,dt$$

$\Leftrightarrow$

(1.1.4) $\quad \psi(s) - \lambda\int_a^s K(s,t)\psi(t)\,dt = 0 \quad \forall s \in [a,b]$

Os. että $\psi = 0$, eli $\varphi_1 = \varphi_2$.

Olk.

$$L = \sup_{[a,b]} |\psi(s)|, \quad N = \sup_{[a,b]\times[a,b]} |K(s,t)|$$

Tällöin

$$|\psi(s)| \le \lambda\int_a^s |K(s,t)|\underbrace{|\psi(t)|}_{\le L}\,dt \le \lambda N L\,(s-a)$$

Iteroidaan:

$$|\psi(s)| \le \lambda\int_a^s |K(s,t)|\underbrace{|\psi(t)|}_{\le \lambda N L(t-a)}\,dt \le LN^2\lambda^2(s-a)^2/2$$

ja induktiolla

$$|\psi(s)| \le LN^n\lambda^n(s-a)^n/n! \quad \forall n \in \mathbb{N} .$$

$$\xrightarrow[n\to\infty]{} 0 \quad \forall s \Rightarrow \psi(s) \equiv 0 . \ \square$$

<u>Esim. 1.1.3.</u> Tarkastellaan Volterra-yhtälöä

$$(1.1.5)\quad \varphi(s) = \sin s + \int_0^s (s-t)\varphi(t)\,dt, \quad s \in [0, 2\pi].$$

Haluamme nähdä kuinka Lauseen 1.1.1 iteratio konvergoi kohti ratkaisua. Etsitään ensin ratkaisu. Tämä onnistuu helpoiten palauttamalla (1.1.5) diff. yhtälön alkuarvo-ongelmaksi.

Derivoidaan (1.1.5):

$$(1.1.6)\quad \varphi'(s) = \cos s + (s-t)\varphi(t)\Big|_{t=s} + \int_0^s \frac{\partial(s-t)}{\partial s}\varphi(t)\,dt$$
$$= \cos s + \int_0^s \varphi(t)\,dt$$

Ja vielä toisen kerran:

$$\varphi''(s) = -\sin s + \varphi(s)$$

eli φ tot. dy:n

$$(1.1.7)\quad \varphi'' - \varphi + \sin s = 0$$

Edelleen (1.1.5) $\Rightarrow$

$$(1.1.8)\quad \varphi(0) = \sin 0 + 0 = 0$$

(1.1.6) $\Rightarrow$

$$(1.1.9)\quad \varphi'(0) = 1$$

Nyt (pikakertaus tavallisista DY:stä) yht. $\varphi'' - \varphi = 0$ yleinen ratkaisu on

$$\varphi_0(s) = C_1 e^s + C_2 e^{-s}$$

Haetaan epähomog. yht. ratkaisua yritteellä

$$\varphi(s) = A \sin s + B \cos s$$
$$\varphi'(s) = A \cos s - B \sin s$$
$$\varphi''(s) = -A \sin s - B \cos s$$

eli

$$\varphi'' - \varphi = -2A \sin s - 2B \cos s = -\sin s$$

$\Leftrightarrow \quad A = 1/2, \; B = 0.$

Siis (1.1.7):n yleinen ratkaisu on

$$\varphi(s) = C_1 e^s + C_2 e^{-s} + \frac{1}{2}\sin s$$

Nyt

$$0 = \varphi(0) = C_1 + C_2 + \frac{1}{2}\cdot 0 \Leftrightarrow C_2 = -C_1$$

$$1 = \varphi'(0) = C_1 - C_2 + \frac{1}{2} = 2C_1 + \frac{1}{2} \Rightarrow C_1 = 1/4 = -C_2$$

$\therefore$ (1.1.5):n ratkaisu on ($\Leftarrow$: Perustele / HT)

$$\varphi(s) = \frac{1}{4}(e^s - e^{-s}) + \frac{1}{2}\sin s = \frac{1}{2}(\cosh s + \sin s).$$

Tutkitaan nyt iteration alkua: $(\lambda = 1)$

$$\varphi(s) = \sum_{n=0}^{\infty} (-1)^n \varphi_n(s), \qquad K(s,t) = s - t$$

missä

$$\varphi_0(s) = \sin(s)$$

$$\varphi_1(s) = \int_0^s (s-t)\varphi_0(s) = \int_0^s (s-t)\sin t\,dt$$

$$= \Big/_0^s (s-t)\left[-\cos t\right] - \int_0^s (-1)(-\cos t)\,dt$$

$$= s - \int_0^s \cos t\,dt = s - \sin(s) \qquad \Big|\ \varphi_0+\varphi_1 = s$$

$$\varphi_2(s) = \int_0^s (s-t)\left[t - \sin(t)\right]dt$$

$$= \Big/_0^s (s-t)\left[\frac{t^2}{2} + \cos(t)\right] - \int_0^s (-1)\left[\frac{t^2}{2} + \cos t\right]dt$$

$$= -S + \int_0^s \frac{t^2}{2} + \cos t\,dt = -S + \Big/_0^s \frac{t^3}{2\cdot 3} + \sin t = -S + \frac{S^3}{6} + \sin(s)$$

Note: $\varphi_0 + \varphi_1 + \varphi_2 = \frac{S^3}{6} + \sin(s)$

$$\varphi_3(s) = \int_0^s (s-t)\left[-t + \frac{t^3}{6} + \sin(t)\right]dt$$

$$= \Big/_0^s (s-t)\left[-\frac{t^2}{2} + \frac{t^4}{4\cdot 6} - \cos t\right] - \int_0^s (-1)\left[-\frac{t^2}{2} + \frac{t^4}{4\cdot 6} - \cos t\right]dt$$

$$= S + \Big/_0^s \left(-\frac{t^3}{2\cdot 3} + \frac{t^5}{4\cdot 5\cdot 6} - \sin t\right)$$

$$= S - \frac{S^3}{2\cdot 3} + \frac{S^5}{4\cdot 5\cdot 6} - \sin(s)$$

$\therefore$

$$\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 = S + \frac{S^5}{4\cdot 5\cdot 6}$$

jne ...

Verrataan tätä ratkaisua Taylor-kehitelmään:

$$\frac{1}{4}\left(e^{s} - e^{-s}\right) + \frac{1}{2}\sin(s)$$

$$= \frac{1}{4}\left(\cancel{1} + s + \cancel{\frac{s^2}{2!}} + \frac{s^3}{3!} - \cancel{1} - (-s) - \cancel{\frac{(-s)^2}{2}} - \frac{(-s)^3}{3!} + \dots\right) + \frac{1}{2}\left(s - \frac{s^3}{3!} + \dots\right)$$

$$= \frac{1}{4}\left(2s + \frac{2s^3}{3!} + \frac{2s^5}{5!} + \dots\right) + \frac{1}{2}\left(s - \frac{s^3}{3!} + \frac{s^5}{5!}\right) + \dots$$

$$= \frac{1}{2}s + \cancel{\frac{s^3}{2\cdot 3!}} + \frac{s^5}{2\cdot 5!} + \frac{1}{2}s - \cancel{\frac{s^3}{2\cdot 3!}} + \frac{s^5}{2\cdot 5!} + \dots$$

$$= S + \frac{S^5}{5!} + \dots$$

ok ↑ ↖ will be fixed later.

$\therefore$ The convergence is not that immediate ...

1.2. Resolventti-ytimet

Voimme löytää ratkaisulle myös toisen esitystavan, joka on joissain tilanteissa parempi.

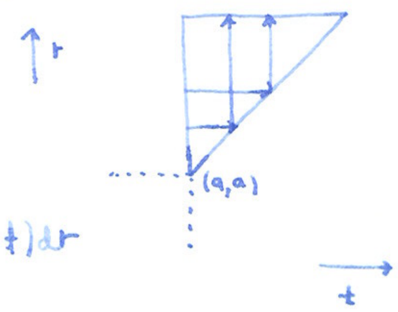
Nyt

$$\varphi_1(s) = \int_a^s K(s,t)f(t)\,dt$$

$$\varphi_2(s) = \int_a^s K(s,r)\varphi_1(r)\,dr = \int_a^s dr \int_a^r K(s,r)K(r,t)f(t)\,dt.$$

Merkitään

$$F(s,r,t) := K(s,r)K(r,t)f(t)$$



Tällöin

$$\int_a^s dr \int_a^r F(s,r,t)\,dt = \int_a^s dt \int_t^s F(s,r,t)\,dr$$

joten

$$\varphi_2(s) = \int_a^s K^{(2)}(s,t) f(t)\,dt,$$

missä

$$K^{(2)}(s,t) = \int_t^s K(s,r)K(r,t)\,dr$$

Voimme jatkaa näin ja määritellä <u>iteroidut</u> <u>ytimet</u>

$$K^{(1)}(s,t) = K(s,t)$$

$$K^{(n)}(s,t) = \int_t^s K(s,r)K^{(n-1)}(r,t)\,dr \quad , n = 2,3,\dots,$$

ja tällöin

$$\varphi_n(s) = \int_a^s K^{(n)}(s,t) f(t)\,dt .$$

Todistus induktiolla kuten tavallista. Jos siis määrittelemme

$$(1.2.1) \quad \Gamma(s,t;\lambda) := K(s,t) + \lambda K^{(2)}(s,t) + \dots + \lambda^{n-1} K^{(n)}(s,t) + \dots$$

saamme ratkaisulle kaavan

$$\varphi(s) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s) = \varphi_0(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\,dt$$

$$= f(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\,dt$$

Nyt

$$|K^{(2)}(s,t)| \le N^2 |s-t|$$

$$|K^{(3)}(s,t)| \le N \int_t^s |K^{(2)}(r,t)|\,dr \le \frac{N^3 (s-t)^2}{2}$$

ja induktiolla

$$|K^{(n)}(s,t)| \le \frac{N^n |s-t|^{n-1}}{(n-1)!},$$

joten (1.2.1) suppenee itseisesti kaikilla $\lambda \in \mathbb{C}$.

<u>Määr. 1.2.1</u> $\Gamma(s,t;\lambda)$ on ydintä $K(s,t)$ vastaava <u>resolventti-ydin</u>.

<u>Lause 1.2.1.</u> $\forall (s,t)$ pätee

$$\Gamma(s,t;\lambda) = K(s,t) + \lambda \int_t^s \Gamma(s,r;\lambda) K(r,t)\,dr$$

$$= K(s,t) + \lambda \int_t^s \Gamma(r,t;\lambda) K(s,r)\,dr$$

"resolventti yhtälö"

Tod. Nyt

$$K^{(n)}(s,t) = \int_t^s K^{(n-1)}(r,t) K(s,r)\,dr \quad , n \ge 2$$

$\Rightarrow$

$$\Gamma(s,t;\lambda) = K(s,t) + \lambda \left[K^{(2)}(s,t) + \dots + \lambda^{n-2} K^{(n)}(s,t) + \dots\right]$$

$$= K(s,t) + \lambda \int_t^s \Big[\underbrace{K^{1}(r,t)K(s,r) + \dots + \lambda^{n-2} K^{(n-1)}(r,t)K(s,r)}_{K(s,r)\left[K^{(1)} + \lambda K^{(2)} + \dots + \lambda^{n-2} K^{(n-1)} + \dots\right]} + \dots\Big]\,dr$$

$$= K(s,t) + \lambda \int_t^s \Gamma(r,t;\lambda) K(s,r)\,dr,$$

eli toinen yhtälö on todistettu.

Palataan nyt lauseeseen

~~Ast~~ $K^{(2)}(s,t) = \int_t^s k(s,r)\,k(r,t)\,dr,$

ja yleisemmin

$$K^{(n)}(s,t) = \int_t^s k(s,r)\,K^{(n-1)}(r,t)\,dr$$

Nyt pätee:

(☺) $K^{(n)}(s,t) = K_{(n)}(s,t)$, $n \geq 2$

nimi

$$K_{(2)}(s,t) = K^{(2)}(s,t) = \int_t^s k(s,r)\,k(r,t)\,dr$$

ja

$$K_{(n)}(s,t) = \int_t^s K_{(n-1)}(s,r)\,k(r,t)\,dr.$$

[Koska (☺) todistus induktiolla ja kirjoittamalla $K^{(n)}$ & $K_{(n)}$:n id. $K^{(n-2)}$ & $K_{(n-2)}$:een avulla.]

Siis samoin

$$\Gamma(s,t;\lambda) = k(s,t) + \lambda \int_t^s \left[K_{(1)}(s,r) + \lambda K_{(2)}(s,r) + \dots\right] k(r,t)\,dr$$

$$= k(s,t) + \lambda \int_t^s \Gamma(s,r;\lambda)\,k(r,t)\,dr$$

joka on toinen yhtälö. □

Let

$$\mathcal{L}\varphi(s) = \varphi(s) - \lambda \int_a^s K(s,t)\,\varphi(t)$$

be an integral operator (Volterra); Under our assumptions on K, this is a bounded linear map $\mathcal{L} : C([a,b]) \to C([a,b])$ (i.e. continuous)

$$\sup_{s\in[a,b]} |\mathcal{L}\varphi(s)| \leq C \sup_{s\in[a,b]} |\varphi(s)|.$$

We have shown that

$$\mathcal{L}\varphi = f \iff \varphi = \mathcal{M}f, \quad \mathcal{M}f(s) = f(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\,dt.$$

i.e. the inverse of a Volterra integral operator is an another Volterra op: $\mathcal{L}^{-1} = \mathcal{M}$, and the resolvent kernel of $\mathcal{L}$ is the kernel of $\mathcal{M}$. Similarly $\mathcal{M}^{-1} = \mathcal{L}$ and hence the resolvent kernel of $\mathcal{M}$ is the kernel of $\mathcal{L}$ i.e. the resolvent kernel of $-\Gamma(s,t;\lambda)$ is $K(s,t)$ itself!

Actually to prove this we need the fact that a (continuous) kernel of a Volterra-op is uniquely determined, but this is trivial ☺; think why!

This follows also directly from the two representations of Prop. 1.2.1.

1.3. Connection to linear diff. eqns

~~Fundamentallösung~~ Ups - Sorry :) Consider an ord. diff. eq.

(1.3.1) $$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = f(x), \quad y^{(k)} = \frac{d^k y}{dx^k}$$

Here $y \in C^n(I)$, $p_k \in C(I)$, $I = (-a,a)$, $a > 0$ an open interval. We want to reduce this to a Volterra eqn of 2nd type.

Def. $$\left\{ g = \int_0^x \dots \int_0^x \varphi \, ds^{(k)} \iff g^{(k)} = \varphi, \quad g^{(\ell)}(0) = 0, \; 0 \le \ell \le k-1 \right.$$

Let

$$z(x) = y^{(n)}(x).$$

Hence

$$y^{(n-1)}(x) = \int_0^x z(s)\,ds + C_1$$

$$\vdots$$

$$y^{(k)}(x) = \underbrace{\int_0^x \dots \int_0^x}_{n-k} z \, dx^{(n-k)} + C_1 \frac{x^{n-k-1}}{(n-k-1)!} + \dots + C_{n-k}$$

$$\vdots$$

$$y(x) = \int_0^x \dots \int_0^x z \, dx + C_1 \frac{x^{n-1}}{(n-1)!} + \dots + C_n$$

Let's insert these to (1.3.1):

$$\left. \begin{aligned} & z + p_1(x) \int_0^x z\,dx + \dots + p_n(x) \int_0^x \dots \int_0^x z \, dx^{(n)} \\ & + \sum_{k=1}^n C_k f_k(x) = f(x), \end{aligned} \right\} \quad (1.3.2)$$

where

$$f_k(x) = p_k(x) + \frac{x}{1!} p_{k+1}(x) + \dots + \frac{x^{n-k}}{(n-k)!} p_n(x)$$

Now we use a little lemma:

Lemma 1.3.1 $\forall$ cont φ

$$\int_0^x \dots \int_0^x \varphi(s)\,ds^{(n)} = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)\,ds$$

Pf. Induction on n: trivial for $n=1$: If $h(x) = \int_0^x \varphi(s)\,dx$, then $h(0) = 0$, $h'(x) = \varphi(x)$.

So assume

$$\int_0^x \dots \int_0^x \varphi(s)\,ds^{(n-1)} = \int_0^x \frac{(x-s)^{n-2}}{(n-2)!} \varphi(s)\,ds.$$

Now if $g(x) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)\,ds$, then $g(0) = 0$ and

$$g'(x) = \int_0^x \frac{(x-s)^{n-2}}{(n-2)!} \varphi(s)\,ds = \int_0^x \dots \int_0^x \varphi(s)\,ds^{(n-1)}$$

and the claim follows. □

Hence we can write (1.3.2) as

(1.3.3) $$z(x) + \int_0^x \underbrace{\left(p_1(x) + \dots + p_n(x) \frac{(x-s)^{n-1}}{(n-1)!} \right)}_{=: k(x,s)} \varphi(s)\,ds = f(x) - \sum_{k=1}^n C_k f_k(x)$$

i.e. as a Volterra-eqn of 2nd kind. By the fund. exist. & uniq. thm of ord. diff. eqns (1.3.1) with initial conds

$$\left\{ \begin{aligned} & y^{(n-1)}(0) = C_1 \\ & \vdots \\ & y(0) = C_n \end{aligned} \right.$$

has a unique sol.; by above this sol. satisfies (1.3.3) and since (1.3.3) is uniquely solvable in $C(I)$, these sols must coincide.

1.4 Singulaariset ytimet

Jatketaan 2:n kertaluvun Volterran yht.

(1.4.1) $$f(s) - \lambda \int_0^x k(s,t)\varphi(t)\,dt = f(s), \quad s,t \in [a,b]$$

missä $k(s,t)$ on muotoa

$$k(s,t) = \frac{P(s,t)}{(s-t)^\alpha}, \quad 0<\alpha<1.$$

$$P \in C([a,b]\times[a,b]).$$

Yritetään ymmärtää onko tällä yhtälöllä resolventtiydintä.

Nyt (formaalisti) $$k^{(2)}(s,t) = \int_t^s k(s,r)k(r,t)\,dr = \int_t^s \frac{P(s,r)P(r,t)}{(s-r)^\alpha (r-t)^\alpha}\,dr$$

Olkoon

$$r = t + (s-t)w \qquad \left(eli \quad w = \frac{r-t}{s-t}, \quad t<r<s\right.$$
$$dr = (s-t)dw \qquad \left. jolta \quad 0 \le w \le 1 \right)$$

Tällöin

$$k^{(2)}(s,t) = \int_0^1 \frac{P(s,t+(s-t)w)P(t+(s-t)w,t)}{[(s-t)(1-w)]^\alpha [(s-t)w]^\alpha}(s-t)\,dw$$

$$= (s-t)^{1-2\alpha}\int_0^1 \frac{P(s,t+(s-t)w)P(t+(s-t)w,t)}{(1-w)^\alpha w^\alpha}\,dw$$

$$=: (s-t)^{1-2\alpha} Q^{(2)}(s,t)$$

[Ups... forgot to write in English...] when $Q^{(2)}(s,t)$ is continuous. We can actually prove:

Lemma 1.4.1. The iterated kernels

$$K^{(1)}(s,t) = K(s,t)$$

$$\vdots$$

$$K^{(n)}(s,t) = \int_t^s K(s,r)K^{(n-1)}(r,t)\,dr$$

are well def and are of the form

$$K^{(n)}(s,t) = \frac{Q^{(n)}(s,t)}{(s-t)^{n\alpha-(n-1)}}, \quad Q^{(n)} \text{ cont.}$$

Pf. By induction on n: We've seen the claim for $n=1,2$.

Assume

$$K^{(n-1)}(s,t) = \frac{Q^{(n-1)}(s,t)}{(s-t)^{(n-1)\alpha-(n-2)}}, \quad Q^{(n-1)} \text{ cont.}$$

Then

$$K^{(n)}(s,t) = \int_t^s K(s,r)K^{(n-1)}(r,t)\,dr$$

$$= \int_t^s \frac{P(s,r)}{(s-r)^\alpha}\frac{Q^{(n-1)}(r,t)}{(r-t)^{(n-1)\alpha-(n-2)}}\,dr \qquad \text{Again } w = \frac{r-t}{s-t}$$

$$= \int_0^1 \frac{P(s,t+(s-t)w)\,Q^{(n-1)}(t+(s-t)w,t)}{(s-t)^{\alpha+(n-1)\alpha-(n-2)}(1-w)^\alpha w^{(n-1)\alpha-(n-2)}}(s-t)\,dw$$

$$= (s-t)^{n-1-n\alpha}\int_0^1 \frac{P(s,t+(s-t)w)\,Q^{(n-1)}(t+(s-t)w,t)}{(1-w)^\alpha \, w^{\underbrace{(n-1)\alpha+2-n}_{<n-1+2-n=1 \text{ since } \alpha<1}}}\,dw$$

So $K^{(n)}(s,t)$ is well def.

and

$$Q^{(n)}(s,t) := \int_0^1 \frac{P(s,t+(s-t)w)\,Q^{(n-1)}((t+(s-t)w,t)}{(1-w)^\alpha w^{(n-1)\alpha+2-n}}\,dw$$

is continuous. □

Choose $n_0 > 0$ s.t. $n \ge n_0 \Rightarrow n\alpha-(n-1) < 0$.

Then $K^{(n)}(s,t)$ are continuous, $n \ge n_0$ and we can proceed as before.

Especially, for $n > n_0$

$$K^{(n)}(s,t) = \int_t^s K(s,r)\underbrace{K^{(n-1)}(r,t)}_{\text{cont}}\,dr$$

and by the argument in 1.2 the series

$$\sum_{n \geq n_0} \lambda^{n-1} K^{(n)}(s,t)$$

converges absolutely in $[a,b]\times[a,b]$ and hence the resolvent kernel is given by

$$(1.4.2)\quad \Gamma(s,t;\lambda) = K(s,t) + \lambda K^{(2)}(s,t) + \dots + \lambda^{n_0-2} K^{(n_0-1)}(s,t) + \sum_{n \geq n_0} \lambda^{n-1} K^{(n)}(s,t).$$

We still need to show that Γ in (1.4.2) is a true resolvent kernel in the sense that it gives rise to sols of (1.4.1)

Prop 1.4.2. Let $\varphi_1(s) = \int_a^s K(s,t) f(t)\,dt$, $\varphi_0(s) = f(s)$

$$\varphi_n(s) = \int_a^s K^{(n)}(s,t) f(t)\,dt.$$

Then $\varphi(s) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(s)$ converges absolutely & unif. on $[a,b]$, φ is a solution of (1.4.1) and

$$(1.4.3)\quad \varphi(s) = f(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\,dt.$$

Pf. Let

$$M = \sup_{[a,b]} |f(s)|,$$

$$K^{(n)}(s,t) = \frac{Q^{(n)}(s,t)}{(s-t)^{n\alpha-(n-1)}}, \quad Q^{(n)} \text{ cont}$$

and for $n < n_0$ let

$$C_n = \sup_{[a,b]^2} |Q^{(n)}(s,t)|.$$

Thus, for $n \leq n_0$,

$$|\varphi_n(s)| \leq M C_n \int_a^s (s-t)^{n-1-n\alpha}\,dt = \frac{M C_n (s-a)^{n-n\alpha}}{n(1-\alpha)}$$

so φ_n might not be bnd on $[a,b]$.

However, when $n \geq n_0$, $n\alpha - (n-1) < 0 \iff n - n\alpha > 1$ so φ_{n_0} is cont. From n_0 onwards the iteration converges abs. as before and (1.4.3) holds by the same argument. □

Prop. 1.4.3. Eq. (1.4.1) has a unique continuous solution.

Pf. As before it is enough to prove that

$$\psi(s) = \lambda \int_a^s K(s,t)\psi(t)\,dt, \quad \psi \in C([a,b]) \implies \psi = 0$$

Then

$$\psi(s) = \lambda \int_a^s K(s,r)\lambda \int_a^r K(r,t)\psi(t)\,dt\,dr = \lambda^2 \int_a^s K^{(2)}(s,t)\psi(t)\,dt$$

$$\vdots$$

$$\psi(s) = \lambda^n \int_a^s K^{(n)}(s,t)\psi(t)\,dt.$$

For n large enough $K^{(n)}$ is cont $\overset{\text{L. 1.1.2}}{\implies} \psi = 0$. □

1.5. Equations of 1st kind

Consider an eqn (Volterra eqn of 1st kind)

$$(1.5.1)\quad \cancel{\varphi(s)=} \int_a^s K(s,t)\varphi(t)\,dt = f(s), \quad K \text{ cont}, \; f, \varphi \in C([a,b]).$$

So we want to solve φ given f. How to proceed?

A natural idea is to try to reduce this to an eqn of 2nd kind by differentiating: Assume $\partial K/\partial s$ is continuous. ($\implies f \in C^1([a,b])$!

Differentiating (1.5.1) we get

$$(1.5.2)\quad K(s,s)\varphi(s) + \int_a^s \frac{\partial K(s,t)}{\partial s}\varphi(t)\,dt = f'(s)$$

If $K(s,s) \neq 0$, $s \in [a,b]$, we can write this as

$$(1.5.3)\quad \varphi(s) + \int_a^s \tilde{K}(s,t)\varphi(t)\,dt = f'(s)/K(s,s)$$

This is a Volterra eqn of 2^{nd} kind and has a unique solution $\varphi \in C([a,b])$.

Prop. 1.5.1. Assume $K, \partial K/\partial s$ are continuous, $f \in C^1([a,b])$. If $f(a) = 0$, $K(s,s) \neq 0 \;\forall s$, then (1.5.1) has a unique sol $\varphi \in C([a,b])$ and it solves (1.5.3).

Pf. Let φ be the unique sol of (1.5.3):

$$\varphi(s) + \frac{1}{K(s,s)}\int_a^s \frac{\partial K(s,t)}{\partial s}\varphi(t)\,dt = f'(s)/K(s,s)$$

$K(s,s) \neq 0$ $\Leftrightarrow$

$$\underbrace{K(s,s)\varphi(s) + \int_a^s \frac{\partial K(s,t)}{\partial s}\varphi(t)\,dt}_{= \frac{d}{ds}\int_a^s K(s,t)\varphi(t)\,dt} = f'(s)$$

Hence

$$\int_a^s K(s,t)\varphi(t)\,dt = f(s) + C, \quad a \le s \le b,$$

and

$$0 = f(a) + C \overset{f(a)=0}{\Rightarrow} C = 0$$

i.e. (1.5.1) holds. If φ solves (1.5.1) with $f = 0$ then φ solves (1.5.3) with $f'/K(s,s) = 0$, hence $\varphi = 0$. $\square$

So even in this trivial case for (1.5.1) to be solvable, f has to satisfy some (obvious) compatibility conds:

$$f \in C^1([a,b]), \quad f(a) = 0.$$

Luckily these are also sufficient in case $K(s,s) \neq 0$.

If $K(s,s) \equiv 0$ then of course above does not work. We can however differentiate eqn

$$(1.5.4)\quad \int_a^s \frac{\partial K(s,t)}{\partial s}\varphi(t)\,dt = f'(s)$$

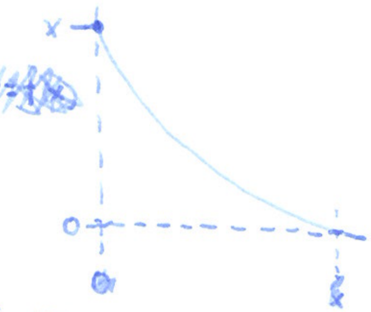
assuming $\partial^2 K/\partial s^2$ is cont. Note that (1.5.4) $\Rightarrow$ $\left\{\begin{array}{l} f'(a) = 0 \\ f \in C^2([a,b]) \end{array}\right.$

If $\partial K(s,s)/\partial s \neq 0$, $a \le s \le b$ this again reduces to a Volterra eqn of 2^{nd} kind. We leave the formulation & pf of a solvability result in this case as an exercise.

1.6. Abel's integral eqn.

Consider the following (classical) problem in ~~mechanics~~ classical mechanics:

Assume a point moves under the influence of gravity along a given curve



Let $f(x)$ be the time required for the point to move from height $x > 0$ to $x = 0$. [Determine the equation of the curve.]

Let $y = F(x)$ be the eqn of the curve. When point moves from height x to ~~pt. at~~ height $x-t$, for veloc. we have

$$v(t) = \sqrt{2g(x-t)} \qquad \Big| \ \Delta E = mg(x-t) = \tfrac{1}{2} m v(t)^2$$

and

$$d\ell = \sqrt{1+F'(t)^2}\, dt,$$

Hence

$$v(t) = \frac{d\ell}{dt} \Rightarrow dt = \frac{d\ell}{v(t)} \Rightarrow t = f(x) = \int_0^x \frac{\sqrt{1+F'(t)^2}\, dt}{\sqrt{2g(x-t)}}$$

This is a Volterra eqn of 1st kind but with singular kernel. This is an example of a so called Abel's eqn.

Consider

(1.6.1) $$K(s,t) = \frac{G(s,t)}{(s-t)^{\alpha}}, \qquad 0<\alpha<1, \ G \text{ cont.}$$
$G(s,s) \neq 0$ kun $s \neq 0$.

An eqn

(1.6.2) $$\int_a^s K(s,t)\varphi(t)\,dt = f(s), \qquad a \le s \le b,$$

with K given by (1.6.1) is a generalized Abel integral eqn.

Now multiply (1.6.2) by $1/(x-s)^{1-\alpha}$:

$$\frac{1}{(x-s)^{1-\alpha}} \int_a^s \frac{G(s,t)}{(s-t)^{\alpha}} \varphi(t) = \frac{1}{(x-s)^{1-\alpha}} f(s), \quad a < s < x,$$

and integrate from $a \to x$:

$$\int_a^x \frac{1}{(x-s)^{1-\alpha}} \left[\int_a^s \frac{G(s,t)}{(s-t)^{\alpha}} \varphi(t)\right] ds = \int_a^x \frac{1}{(x-s)^{1-\alpha}} f(s)\,ds$$

$\Leftrightarrow$

$$\int_a^x \varphi(t)\,dt \left\{ \int_t^x \frac{G(s,t)}{(x-s)^{1-\alpha}(s-t)^{\alpha}} ds \right\} = \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} f(s)$$

Let

$$K_1(x,t) := \int_t^x \frac{G(s,t)}{(x-s)^{1-\alpha}(s-t)^{\alpha}} ds \qquad s = t + r(x-t)$$

$$= \int_0^1 \frac{G(t+r(x-t),t)}{(1-r)^{1-\alpha} r^{\alpha}} dr$$

and

$$F(x) := \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds$$

$$= (x-a)^{\alpha} \int_0^1 \frac{f(a+r(x-a))\,dr}{(1-r)^{1-\alpha}}, \quad s = a + r(x-a)$$

Then K_1 & F are cont. and we have

(1.6.3) $$\int_a^x K_1(x,t)\varphi(t)\,dt = F(x);$$

a Volterra eqn of the 1st kind.

Lemma 1.6.1 If f' is cont, then $F \in C^1((a,b)) \cap C([a,b])$.

Pf. For $h \neq 0$, $|h|$ small enough, $x \in (a,b)$,

$$\frac{F(x+h)-F(x)}{h} = (x+h-a)^{\alpha} \int_0^1 \frac{f(a+r[x+h-a]) - f(a+r[x-a])}{h(1-r)^{1-\alpha}} dr$$
$$+ \frac{(x+h-a)^{\alpha} - (x-a)^{\alpha}}{h} \int_0^1 \frac{f(a+r[x-a])}{(1-r)^{1-\alpha}} dr \xrightarrow[h \to 0]{}$$

$$(x-a)^{\alpha}\int_0^1 \frac{f'(a+r[x-a])}{(1-r)^{1-\alpha}}\,dr + \alpha(x-a)^{\alpha-1}\int_0^1 \frac{f(a+r[x-a])}{(1-r)^{1-\alpha}}\,dr$$

$\square$

<u>Lemma 1.6.2.</u> $\Big\{$ Ass $\partial G/\partial s$ is cont. Then $K_1(x,x)\neq 0$ and $\partial K_1/\partial x$ is continuous.

Pf. $K_1(x,x) = G(x,x)\int_0^1 \frac{dr}{(1-r)^{1-\alpha}r^{\alpha}} \neq 0$

Also

$$\frac{\partial K_1(x,t)}{\partial x} = \int_0^1 \frac{\partial G(t+r[x-t],t)}{\partial x}(1-r)^{\alpha-1}r^{-\alpha}\,dr$$

is cont. $\square$

Hence (1.6.1) is always uniquely solvable. We have

<u>Prop. 1.6.3.</u> Jos $\varphi\in C([a,b])$ tot. yht. (1.6.3) niin se on ... Hups... back to English ..., then it also satisfies (1.6.2).

Pf. Let

$$h(s) = \int_a^s \frac{G(s,t)}{(s-t)^{\alpha}}\varphi(t)\,dt - f(s).$$

By above

$$\int_a^x \frac{h(s)}{(x-s)^{1-\alpha}}\,ds \equiv 0, \quad a<x<b.$$

and hence

$$0 = \int_a^y \frac{1}{(y-x)^{\alpha}}\left[\int_a^x \frac{h(s)}{(x-s)^{1-\alpha}}\,ds\right]dx = \int_a^y h(s)\,ds\left[\int_0^1 \frac{dr}{(1-r)^{\alpha}r^{1-\alpha}}\right]$$

Now (use residues for example; Ex.)

$$\int_0^1 \frac{dr}{(1-r)^{\alpha}r^{1-\alpha}} = \frac{\pi}{\sin\pi\alpha} \neq 0 \text{ when } 0<\alpha<1$$

and hence

$$\int_a^y h(s)\,ds = 0 \ \forall\, y \Rightarrow h = 0. \ \square$$

II Hilbert spaces and cpt operators

2.1. Hilbert-spaces

Def. 2.1.1 A (complex) vector space X is an <u>inner product space</u> if there is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfying

(IP1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{C},\ x, y, z \in X$

(IP2) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$

(IP3) $\langle x, x \rangle \geq 0 \quad \forall x \in X$

(IP4) $\langle x, x \rangle = 0 \iff x = 0$.

So $\langle \cdot, \cdot \rangle$ is linear in the 1^{st} variable, conjugate linear in 2^{nd}.

So classical examples are:

Ex. 2.1.2. i) $\mathbb{C}^n = \{(z_1, \dots, z_n) ; z_k \in \mathbb{C}\}$. Then the inner product is just the euclidean inner product

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}, \quad z = (z_1, \dots, z_n),\ w = (w_1, \dots, w_n)$$

ii) Let $X = C([a,b])$, and define

$$\langle f, g \rangle = \int_a^b f \overline{g}\, dx .$$

Then this is an inner product on $C([a,b])$.

iii) Let $X = \ell^2(\mathbb{C}) = \{(z_n)_{n=1}^{\infty} ; \sum_{n=1}^{\infty} |z_n|^2 < \infty\}$.
Then $\ell^2(\mathbb{C})$ is an inner product space when

$$\langle z, w \rangle = \sum_{i=1}^{\infty} z_i \overline{w_i}, \quad z = (z_1, z_2, \dots),\ w = (w_1, w_2, \dots)$$

Show that $\langle z, w \rangle$ is well def. in ℓ^2 and defines an inner prod.

If X is an inner-product space, then it is a normed space in a canonical way:

define $\|x\| = \langle x, x \rangle^{1/2}$, $x \in X$.

Then $\|\cdot\|$ is a norm. Proof follows as in the Euclidean case once the following is known:

Prop. 2.1.3. (Cauchy-Schwartz ineq.) If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space then

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \forall x, y \in X .$$

Pf. Choose $\omega \in \mathbb{C}$, $|\omega| = 1$ s.t.

$$\langle x, \omega y \rangle = |\langle x, y \rangle| = \langle \omega y, x \rangle .$$

Then $\forall t \in \mathbb{R}$,

$$0 \leq \langle x + t\omega y, x + t\omega y \rangle = \|x\|^2 + t^2 \underbrace{\|y\|^2}_{\text{since } \|y\| = \|\omega y\|} + t(\langle x, \omega y \rangle + \langle \omega y, x \rangle)$$

$$= \|x\|^2 + 2t |\langle x, y \rangle| + t^2 \|y\|^2 .$$

Hence discrim. $\leq 0 \iff |\langle x, y \rangle| \leq \|x\| \|y\|$. $\square$

All the norm-properties for $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ except the triangle ineq. are trivial. This follows now from 2.1.3:

$$\|x+y\|^2 = |\langle x+y, x+y \rangle| = |\|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle|$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

proving the Δ-ineq.

For us having an inner-product is not enough. We need to know the three spaces are also complete normed spaces:

Def. 2.1.4 An inner product space $(H, \langle \cdot , \cdot \rangle)$ which is complete w.r.t. metric $\| \cdot \| = \langle \cdot , \cdot \rangle^{1/2}$ defined by the inner product is a Hilbert-space.

[Complete here means that all Cauchy-sequences in H converge to an element of H]

Ex. 2.1.5 i) $\mathbb{C}^n$ and $\ell^2(\mathbb{C})$ are complete $\leftarrow$ [Funct. anal. course

ii) $C([a,b])$ is not complete w.r.t. to norm

$$\|f\|^2 = \int_a^b |f|^2 dx \ ;$$

Namely, let $a = -1$, $b = 1$ and

$$f_n(x) = \begin{cases} 0, & 0 \le x < 0 \\ nx, & 0 \le x \le 1/n \\ 1, & 1/n < x \le 1 \end{cases}$$

Then f_n is continuous, and if $H(x) = \chi_{[0,1]}(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$ we have

$$\int_{-1}^{1} |f_n - H|^2 dx = \int_0^{1/n} |nx-1|^2 = \int_0^{1/n} (n^2x^2 - 2nx + 1) dx$$

$$= \Big/_0^{1/n} \frac{n^2x^3}{3} - nx^2 + x = \frac{1}{3n} - \frac{1}{n} + \frac{1}{n} = \frac{1}{3n} \xrightarrow{n\to\infty} 0$$

hence $\|f_n - H\| \to 0$, but $H \notin C([-1,1])$.

Note that $C([a,b])$ is complete w.r.t. to ~~max~~ sup-norm

$$\|f\|_{sup} = \sup_{a \le x \le b} |f(x)| ,$$

but $\| \cdot \|_{sup} \ne \| \cdot \|$ and there is no inner product inducing $\| \cdot \|_{sup}$ - norm (Exercises ...)

If one wants ~~consider~~ to consider $C([a,b])$ with ~~an~~ the inner-product norm one is led to L^2-spaces; (see Reallanal. I)

Def. 2.1.6 If $\Omega \subseteq \mathbb{R}^n$, then

$$L^2(\Omega) = \{ [f] ;\ f : \Omega \to \mathbb{C} ,\ \int_\Omega |f|^2 dx < \infty \}$$

Lebesgue measure

Here $[f]$ is the equivalence class of all meas. functions g s.t. $g = f$ a.e. We define the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f \bar{g} \, dx \qquad \text{(well def. in equiv. classes)}$$

This is a Hilbert-space with norm

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f|^2 dx \right)^{1/2} .$$

2.2. Bounded linear ops in Hilbert-spaces

Recall that a linear op $A : X \to Y$, X, Y normed spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ is continuous iff it is bounded i.e. $\exists$ const. $M > 0$ s.t.

(2.2.1) $$\|Ax\|_Y \le M \|x\|_X , \ \forall x \in X .$$

[Proof of this: (2.2.1) $\Rightarrow$ A continuous at 0, since it is linear it is continuous everywhere.

$\Leftarrow$: Assume A is not continuous at x_0. Since A linear we may assume $x_0 = 0$.

If (2.2.1) does not hold, $\forall n \ \exists x_n \in X$ s.t.

$$\|Ax_n\|_Y \ge n \|x_n\|_X .$$