

Now we only need to show that $\|A_n - A\| \rightarrow 0, n \rightarrow \infty$. Now let $m, n > \frac{1}{\epsilon}$.
 Then

$$\|A - A_n\| \times \|y\| = \lim_{m \rightarrow \infty} \|A_m x - A_n x\|_y \leq \epsilon \|x\|_X \quad \forall x \in X$$

$$\Rightarrow \|A - A_n\| < \epsilon. \text{ Hence } \|A - A_n\| \xrightarrow{n \rightarrow \infty} 0.$$

If $X = H_1$ and $Y = H_2$ are Hilbert spaces all this holds and esp. $\mathcal{L}(H_1, H_2)$ is a complete normed space i.e. a Banach space!

Note: $\mathcal{L}(H_1, H_2)$ is usually not a Hilbert-space even if H_1 and H_2 are. In a Hilbert space there are two very special types of linear ops.

2.2.1. Projections (i.e. ortho-projections)

Recall the followings in \mathbb{R}^n : let $M \subset \mathbb{R}^n$ be a subspace.

Let (m_1, \dots, m_k) be its orthonormal base i.e. $\|m_j\| = 1$ and $m_i \perp m_j$ are orthogonal to each other if $i \neq j$. Then \exists basis $(m_1, \dots, m_k, m_{k+1}, \dots, m_n)$ of whole \mathbb{R}^n that is also orthonormal.

Especially \exists bind map

$$P: \sum_{i=1}^n x_i m_i \rightarrow \sum_{i=1}^k x_i m_i \in M$$

and $M^\perp = \text{span}\{m_{k+1}, \dots, m_n\}$, a map

$$Q: \sum_{i=1}^n x_i m_i \rightarrow \sum_{i=k+1}^n x_i m_i \in M^\perp$$

s.t. $P \perp Q \forall x, x = Px + Qx$

P is the orthogonal proj. to M
 Q is the orthogonal proj. to M^\perp .

Let $X_n = \frac{x_n}{n} \forall x_n \in X$. Then $\|y_n\| = \frac{1}{n} \rightarrow 0, n \rightarrow \infty$
 but

$$\|A y_n\| = \frac{\|A x_n\|}{n} \geq 1$$

so A is not continuous at 0. \square

We denote $\mathcal{L}(X, Y) = \{A; A: X \rightarrow Y \text{ bind. \& linear}\}$

The inf of all M satisfying (2.2.1) is denoted by $\|A\|$ and is the norm of A . This really defines a Banach

norm on $\mathcal{L}(X, Y)$ (X, Y in HNS). Furthermore, $\mathcal{L}(X, Y)$ is complete then also $\mathcal{L}(X, Y)$ is complete w.r.t. operator-norm: Assume $A_n \in \mathcal{L}(X, Y), n \in \mathbb{N}$, is Cauchy i.e.

$$\forall \epsilon, \exists n_\epsilon \text{ s.t. } m, n > n_\epsilon \Rightarrow \|A_m - A_n\| < \epsilon.$$

Then $\forall x \in X$,

$$\|A_m x - A_n x\| \leq \|A_m - A_n\| \|x\| < \epsilon \|x\|,$$

so $(A_m x)$ is a Cauchy-sequence in $Y \Rightarrow \exists$ limit

$$A x := \lim_{n \rightarrow \infty} A_n x.$$

It is easy to see that A is linear. It is also bounded:

Choose $m_0 \text{ s.t. } m, n > m_0 \Rightarrow \|A_m - A_n\| < 1, \forall m, n > m_0$
 then $\forall n > m_0$,

$$\|A_n\| < 1 + \|A_{m_0}\|$$

and then

$$\|A_n x\| \leq (1 + \|A_{m_0}\|) \|x\|_X$$

But $\|A x\|_Y = \lim \|A_n x\|_Y \leq (1 + \|A_{m_0}\|) \|x\|_X$

$\therefore A$ bind.

This is still true in Hilbert-spaces (but usually not in Banach spaces) if we assume M is closed.

Note that in the finite dimensional case all subspaces are autom.-closed. In Hilbert spaces this need not hold (let $M = C([0,1]) \subset L^2([0,1])$).

• First, if $x, y \in H$, H Hilbert, then by def.

$$\begin{cases} x+y & (x \perp y \text{ orthogonal to each other}) \\ \text{iff } \langle x, y \rangle = 0 \end{cases}$$

• If $M \subset X$ is a subspace, then we define the orthogonal complement M^\perp as

$$M^\perp = \{y \in H; \langle x, y \rangle = 0 \forall x \in M\}$$

(even if M is not!))

• Note that M^\perp is always a closed subspace:

if $\alpha, \beta \in \mathbb{C}$, $y_1, y_2 \in M^\perp$ then $\forall x \in M$

$$\langle x, \alpha y_1 + \beta y_2 \rangle = \alpha \langle x, y_1 \rangle + \beta \langle x, y_2 \rangle \underset{=0}{=} 0$$

ie. $\alpha y_1 + \beta y_2 \in M^\perp$.

Also, if $y_n \in M^\perp$, $y_n \rightarrow y \in H$, then $\forall x \in M$:

$$0 = \langle x, y_n \rangle \rightarrow \langle x, y \rangle \Rightarrow \langle x, y \rangle = 0 \text{ i.e. } y \in M^\perp$$

Thm. If M is a closed subspace of H , then \exists unique pair of linear maps $P, Q \leq \mathcal{L}(H)$.

$$P: H \rightarrow M, \quad Q: H \rightarrow M^\perp$$

and

$$(P1) \quad X = Px + Qx \quad \forall x \in H$$

Also

$$(P2) \quad \forall x \in M \Rightarrow Px = x, \quad Qx = 0$$

$$x \in M^\perp \Rightarrow Px = 0, \quad Qx = x$$

$$(P3) \quad \|x - Px\| = \|Qx\| = \inf \{ \|x - y\|; y \in M \}$$

$$(P4) \quad \|x\|^2 = \|Px\|^2 + \|Qx\|^2 \quad (\text{Pythagoras})$$

Pf: Omitted; see for example [Astala-Tylli (Funktionalanalyse), or Rudin, Real & Complex, pp. 34-35 (at least in my edition :)]

2.2.2. Bounded linear functionals

Let's again start from \mathbb{R}^n ; consider a linear map (a functional)

$$\lambda: \mathbb{R}^n \rightarrow \mathbb{R}$$

How could one try to represent λ ? let (e_1, \dots, e_n) be the standard basis of \mathbb{R}^n ; then

$$\lambda(x) = \lambda\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i \lambda(e_i) = \langle x, \tilde{x} \rangle, \quad \tilde{x} = (\lambda(e_1), \dots, \lambda(e_n)).$$

Hence λ is just inner product with \tilde{x}_λ , a fixed element of \mathbb{R}^n !

It is not wise to expect this to be true in the infinite dimensional case; for example if one equips $C([0,1])$ with sup-norm (hence it becomes a Banach space but not a Hilbert space), the space of linear functionals on $([0,1])$ is the space of regular Borel measures on $[0,1]$ - a huge space, much larger than $C([0,1])$.

However on a Hilbert space things work out great:

$\Rightarrow \langle x, y_1 - y_2 \rangle = 0 \quad \forall x \in H$. Choose $x = y_1 - y_2$. Then
 $0 = \langle y_1 - y_2, y_1 - y_2 \rangle = \|y_1 - y_2\|^2 \Leftrightarrow y_1 = y_2. \quad \square$

2.3 Neumann series (Named after a 19th-century German mathematician, Carl Neumann)

Let's get back to -- or at least closer to -- integral equations.
 Let $A \in \mathcal{L}(H, H)$; H Hilbert (actually the inner product plays no role for the main thing; it holds in any Banach space).
 we prove the following:
 given $M \in H$

Thm 2.3.1 Assume $\|A\| < 1$. Then the eqn

$$x - Ax = y$$

has a unique solution $x \in H$ and

$$x = y + Ay + \dots + A^m y + \dots$$

in fact, $I - A$ is invertible in $\mathcal{L}(H)$ and

$$I - A = I + A + \dots + A^m + \dots = \sum_{k=0}^{\infty} A^k$$

and this series converges in operator norm.

Pf. Let $B_{m,n} = \sum_{k=m}^n A^k \quad (m < n)$.

Now $\|B_{m,n}\| \leq \sum_{k=m}^n \|A\|^k = \|A\|^m \sum_{k=0}^{n-m} \|A\|^k = \|A\|^m \frac{1 - \|A\|^{n-m+1}}{1 - \|A\|} \rightarrow 0$
 since $\|A\| < 1$. Hence the partial sums $\sum_{k=0}^n A^k$ form a Cauchy sequence in $\mathcal{L}(H, H)$.

Thm. (Riesz representation thm.) If $\lambda: H \rightarrow \mathbb{C}$ is a bounded linear functional i.e. λ linear and $\exists M > 0$ s.t.

$$|\lambda(x)| \leq M \|x\| \quad \forall x \in H,$$

then \exists unique $y \in H$ s.t.

$$\lambda(x) = \langle x, y \rangle \quad \forall x \in H.$$

Pf. This follows easily from \exists projections. If $\lambda \equiv 0$, then $y = 0$ and this is unique. So assume $\lambda \neq 0$.

Let $M = \{x : \lambda(x) = 0\}$.

M is a closed (since λ cont.) subspace of H . Also,

$M^\perp \neq \emptyset$ (orthogonality proj thm. $\Rightarrow M = H \Rightarrow \lambda = 0$).

Choose $z \in M^\perp, \|z\| = 1$. Let

$$u := \lambda(z)z - \lambda(z)x.$$

Then

$$(i) \quad \lambda(u) = \lambda(x)\lambda(z) - \lambda(z)\lambda(x) = 0 \Rightarrow u \in M.$$

Hence $\langle u, z \rangle = 0$, and

$$\lambda(x) = \lambda(x)\langle z, z \rangle = \langle \lambda(z)z, z \rangle = \|\lambda(z)\|^2 = 1$$

$$(ii) \quad \begin{aligned} &= \langle \lambda(z)x, z \rangle = \lambda(z)\langle x, z \rangle \\ &= \langle x, z\lambda(z) \rangle. \end{aligned}$$

So choosing $y = z\lambda(z)$ we get existence. Uniqueness follows easily: If $\forall x \in H$:

$$\lambda(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$$

and \exists limit $X(H, H) \ni B := I + A + \dots + A^n + \dots$
 Let's prove that $(I-A)B = B(I-A) = I$. Let

$$B_n = I + A + \dots + A^n$$

Then $(I-A)B_n = I - A^{n+1}$

$$B_n(I-A) = I - A^{n+1}$$

so since $\|A\| < 1$, $\|A\|^{n+1} \leq \|A\|^n \rightarrow 0$ and thus

$$I - A \xrightarrow{n \rightarrow \infty} I$$

and $(I-A)B = \lim_n (I-A)B_n = I = \lim_n B_n(I-A) = B(I-A)$

This proves the Thm. \square

["Small perturbations of identity are still invertible" applications to

Before moving to integral eqs, let's prove the following important consequence of Thm. 2.3.1:

Prop. 2.3.2. Let H_1, H_2 be \mathcal{K} (loc). Let

$$\text{Inv}(H_1, H_2) = \{ A \in \mathcal{L}(H_1, H_2) ; \exists B \in \mathcal{L}(H_2, H_1) \text{ s.t.}$$

$$AB = I_{H_2}, BA = I_{H_1} \}$$

The set of invertible elements

Then in the operator norm topology $\text{Inv}(H_1, H_2)$ is an open subset of $\mathcal{L}(H_1, H_2)$.

Pf: Let $A \in \text{Inv}(H_1, H_2)$, then for any $R \in \mathcal{L}(H_1, H_2)$

$$A+R = A(I_{H_1} + A^{-1}R)$$

and when $\|A^{-1}R\| < 1$ is $I_{H_1} + A^{-1}R$ inv, and also $A+R$ as a product of inv. ops. But

$$\|A^{-1}R\| \leq \|A^{-1}\| \|R\|$$

when $\|R\| < (\|A^{-1}\|)^{-1}$

$$\|R\| < (\|A^{-1}\|)^{-1} \Rightarrow A+R \text{ is inv. } \square$$

Let's now apply this to integral ops: 1st we want to estimate the operator norm of an integral operator

$$Kf(x) = \int_{\Omega} K(x,y)f(y)dy, x \in \Omega.$$

Here we assume Ω that K is a mean. fund. on $\Omega \times \Omega$ s.t. functions

$$x \mapsto K(x,y), y \mapsto K(x,y)$$

are resp. end. a.o. xy or a.o. yx .

Prop. 2.3.3 (Schur's lemma).

If $\sup_x \int_{\Omega} |K(x,y)|dy, \sup_y \int_{\Omega} |K(x,y)|dx \leq M < \infty$

then

$$K : L^2(\Omega) \rightarrow L^2(\Omega), \|K\| \leq M.$$

Pf. This is a straightforward application of Cauchy-Schwartz: Let $f \in C_0^\infty(\Omega)$ initially. Then

$$|Kv(x)| \leq \int_{\Omega} |K(x,y)| |v(y)| dy$$

$$\stackrel{C-3}{\leq} \underbrace{\left(\int_{\Omega} |K(x,y)|^{1/2} dy \right)^{1/2} \left(\int_{\Omega} |K(x,y)|^{1/2} |v(y)|^2 dy \right)^{1/2}}_{\leq M^{1/2}}$$

Hence integrating over x ,

$$\begin{aligned} \int_{\Omega} |Kv(x)|^2 dx &\leq M \int_{\Omega \times \Omega} |K(x,y)| |v(y)|^2 dy dx \\ &\leq M \left(\sup_y \int_{\Omega} |K(x,y)| dx \right) \int_{\Omega} |v(y)|^2 dy \\ &\leq M^2 \|v\|_{L^2(\Omega)}^2 \leq M \|v\|_{L^2(\Omega)}^2 \end{aligned}$$

and thus

$$\|Kv\|_{L^2(\Omega)} \leq M \|v\|_{L^2(\Omega)} \quad \square$$

Consider now the integral eqn:

$$(C) \quad f(x) + \int_{\Omega} K(x,y) f(y) dy = g(x),$$

where K is a mean function on $\Omega \times \Omega$.

Prop. 2.3.1. If

$$\sup_x \int_{\Omega} |K(x,y)| dy, \sup_y \int_{\Omega} |K(x,y)| dx < 1,$$

then the integral eqn (C) has a unique solution

$f \in L^2(\Omega)$ for all $g \in L^2(\Omega)$.

Pf. Apply Schur's Lemma let

$$Tf(x) = \int_{\Omega} K(x,y) f(y) dy$$

and the result then follows from Neumann-series.

2.4. Compact operators in Hilbert spaces

The Neumann series was based on idea that "small perturbations of the identity are invertible" and we measured the size of an op using operator norm. There are other ways to control the size of an operator.

Recall that in \mathbb{R}^n (\mathbb{C}^n) compact sets are precisely those which are closed and bounded. This is not true in the infinitely dimensional case; let H be an inf. dimensional Hilbert space; then def. seq. $(u_n)_{n=1}^{\infty}$ of lin. indep. elements. Gram-Schmidt \Rightarrow we may assume $(u_n)_{n=1}^{\infty}$ is orthonormal;

$$\|u_n\| = 1, \quad \langle u_n, u_m \rangle = 0, \quad n \neq m.$$

Note that all

$$u_n \in B = \{x \in H; \|x\| \leq 1\} \text{ which is closed \& bounded.}$$

On a opt set of a metric space (normed spaces are metric) every sequence contains a convergent subsequence.

But for $(u_n)_{n=1}^{\infty}$:

$$\langle u_n, u_m \rangle = 0$$

$$n \neq m \Rightarrow \|u_n - u_m\|^2 = \langle u_n - u_m, u_n - u_m \rangle = \|u_n\|^2 + \|u_m\|^2 = 2$$

$$\Rightarrow \|u_n - u_m\| = \sqrt{2} \quad \therefore (u_n)_{n=1}^{\infty} \text{ is not Cauchy} \Rightarrow \text{does not converge!}$$

Recall that the adjoint A^* of $A \in \mathcal{L}(H_1, H_2)$ is def. by $\langle Ax, y \rangle_{H_2} = \langle x, A^*y \rangle_{H_1} \quad \forall x \in H_1, y \in H_2$.

It is an easy consequence of Riesz-rep that A^* exists & is bounded:

Consider $\lambda_\gamma : H_1 \rightarrow \mathbb{C}, x \mapsto \langle Ax, \gamma \rangle$.
 Then λ_γ is a linear functional of $H_1 \Rightarrow \exists z \in H_1$ s.t. $\lambda_\gamma(z) = \langle z, \gamma \rangle$
 $\langle A^* \gamma, \gamma \rangle = \|\lambda_\gamma\|_{H_1} = \|\lambda_\gamma\|$

Def. $A^*_\gamma = z$. Since $|\langle Ax, \gamma \rangle| \leq \|A\| \|x\| \| \gamma \| \quad \forall x \in H_1$
 $\Rightarrow \| \lambda_\gamma \| \leq \|A\| \| \gamma \|$ and thus $\|A^*_\gamma\|_{H_2} \leq \|A\| \| \gamma \|$ and A^* is bounded.

Linearity follows from $\langle x, A^*(\alpha \gamma_1 + \beta \gamma_2) \rangle = \bar{\alpha} \langle Ax, \gamma_1 \rangle + \bar{\beta} \langle Ax, \gamma_2 \rangle = \langle x, \alpha A^* \gamma_1 + \beta A^* \gamma_2 \rangle$.

So now let's prove that K^* is cpt if K is cpt.
 Pf. We need the following fundamental result of functional analysis: If $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of a Hilbert space H , it contains a weakly convergent

So give: $\{\text{cpt sets}\} \subseteq \{\text{bd. \& closed}\}$.

Def. 2.4.1 A bounded op $K : H_1 \rightarrow H_2$ is compact if it maps every bounded set U in H_1 (Hilbert or a precompact set) in H_2 i.e. if $\overline{K(U)}$ is compact $\forall U \subset H_1$ bounded. $\mathcal{K}(H_1, H_2) = \{K \in \mathcal{L}(H_1, H_2) : K \text{ cpt}\}$

Rem. i) All linear maps to finite dimensional spaces are compact.
 ii) Let $B_{H_1} = \{x \in H_1 : \|x\| = 1\}$ be the unit ball of H_1 . Then it is enough to assume that $\overline{K(B_{H_1})}$ is compact.

We'll see in HWs the following:

- a) If $A, B : H_1 \rightarrow H_2$ are cpt then $A+B$ is cpt
- b) If $K : H_1 \rightarrow H_2$ is cpt then AK is cpt $\forall A \in \mathcal{L}(H_2, H)$
 $KB \xrightarrow{\text{cpt}} B \in \mathcal{L}(H_1, H_1)$
- c) If $K_n \in \mathcal{K}(H_1, H_2)$ and $K \in \mathcal{L}(H_1, H_2), \|K_n - K\| \xrightarrow{n \rightarrow \infty} 0$ then $K \in \mathcal{K}(H_1, H_2)$.

Now we'll prove the following:
 Prop. 2.4.2 if $K \in \mathcal{K}(H_1, H_2)$ is cpt, then $K^* \in \mathcal{K}(H_2, H_1)$ is also cpt.

2.5 Integral ops with weakly singular kernels

Let's now study the compactness properties of

$$K(x,y) = \int_{\Omega} k(x,y) \omega(y) dy, \quad \Omega \subseteq \mathbb{R}^n, \quad x, y \in \Omega.$$

We will use the following result:

Thm 2.5.1 Let $\mathcal{F}(H_1, H_2) = \{F \in \mathcal{L}(H_1, H_2) : F \text{ has fin. dimensional image-span}\}$. Then (H_1, H_2) Hilbert-sp.

$$\overline{\mathcal{F}}(H_1, H_2) = \mathcal{K}(H_1, H_2)$$

where the closure is w.r.t. operator-norm.

For the pf, see [Yosida; Functional Analysis].

Also $\overline{\mathcal{F}} \subset \mathcal{K}$

is true also in Banach-spaces, but the inclusion

$$\mathcal{K} \subset \overline{\mathcal{F}}$$

is not always. (Also $\overline{\mathcal{F}}$ is easy to prove; see HWB3)

Now we can prove:

Prop 2.5.2 If Ω bnd and $K \in C(\overline{\Omega} \times \overline{\Omega})$ then $\mathcal{K} \in \mathcal{K}(L^2(\Omega), L^2(\Omega))$.

Pf. To keep notations simple, we consider only the case when $n=1$, $\Omega = (a, b) \subset \mathbb{R}$. Given $n \in \mathbb{N}$, let

subsequence $(u_{n_k})_{k \in \mathbb{N}}$ i.e. sequence

$\langle u_{n_k}, y \rangle_{k \in \mathbb{N}} \forall y \in H$ [Pf: True even for Banach spaces & bnd linear funct. on them; see lectures on Funct. Analysis by Astash-Tyrt] \downarrow

Assume now that $(y_n)_{n \in \mathbb{N}}$ is a bnd sequence in H_2 ; We want to prove that $(K^* y_n)_{n \in \mathbb{N}}$ contains a convergent subsequence. Let

$$y_{n_k} \rightarrow y \text{ weakly in } H_2.$$

Then

$$\begin{aligned} \|K^* y_{n_k} - K^* y\|_{H_1}^2 &= \langle K^* y_{n_k} - K^* y, K^* y_{n_k} - K^* y \rangle \\ &= \langle KK^* y_{n_k} - KK^* y, y_{n_k} - y \rangle. \end{aligned}$$

Now KK^* cpt, (y_n) bnd, so passing to a subsequence of (y_{n_k}) if necessary we may assume that

$$\|KK^*(y_{n_k} - y)\|_{H_2} \rightarrow 0,$$

but since (y_{n_k}) bnd we have by (C)

$$\|K^* y_{n_k} - K^* y\|_{H_1}^2 \leq C \|KK^*(y_{n_k} - y)\|_{H_2} \xrightarrow{n_k \rightarrow \infty} 0. \quad \square$$

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$$a_{k,l} = a + \frac{k(b-a)}{n}, \quad k=0, \dots, m$$

$$\Omega_{k,l} = [a_k, a_{k+1}] \times [b_l, b_{l+1}], \quad k, l=0, \dots, m-1$$

and choose $x_{k,l} \in \Omega_{k,l}$.

Then K unif. cont. in $\bar{\Omega} \times \bar{\Omega} \Rightarrow$

$$\sup_x \int_{\bar{\Omega}} |(K - K^{(n)})(x, y)| dx, \quad \sup_{\bar{\Omega}} \int_{\bar{\Omega}} |(K - K^{(n)})(x, y)| dx \rightarrow 0$$

where

$$K^{(n)}(x, y) = \sum_{k,l=0}^{m-1} K(x_{k,l}) \chi_{\Omega_{k,l}}(x, y).$$

Hence, if

$$K f(x) = \int_{\bar{\Omega}} K(x, y) f(y) dy,$$

Schem $\Rightarrow K^{(n)} f(x) \rightarrow K$ in $\mathcal{L}(\bar{\Omega}, \bar{\Omega})$. But

$$K^{(n)} f(x) = \sum_{k,l=0}^{m-1} K(x_{k,l}) \int_{\Omega_{k,l}} \chi_{\Omega_{k,l}}(x, y) f(y) dy$$

$$\Rightarrow \sum_{k,l=0}^{m-1} K(x_{k,l}) \chi_{[a_k, a_{k+1}]}(x) \int_{a_l}^{a_{l+1}} f(y) dy$$

$$K^{(n)} f \in \text{span}\{\chi_{[a_0, a_1]}, \dots, \chi_{[a_{m-1}, a_m]}\}$$

hence $K^{(n)}$ opt \mathcal{L} K also.

The case of $\dim \Omega = n$ follows similarly approximating \mathcal{R} by a linear combination of characteristic-functions of cubes, we leave this to you. \square

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Consider now a singular kernel of type

$$(WS) \quad K(x, y) = \frac{Q(x, y)}{|x-y|^\alpha}, \quad \alpha < n, \quad Q \in C(\bar{\Omega} \times \bar{\Omega})$$

and $\Omega \subseteq \mathbb{R}^n$ bnd; a kernel of this type is called weakly singular; if $\alpha = n$, K is an example of so called Coulomb-Zygmund kernel; if $\alpha > n$ we allow for $\Omega = \mathbb{R}^n$ and if $\alpha > n$, then the kernel is hypersingular.

We will consider only weakly singular kernels.

Prop. 2.5.3 If $\Omega \subseteq \mathbb{R}^n$ bnd and K is weakly singular, then $K: \bar{L}^2(\Omega) \rightarrow \bar{L}^2(\Omega)$ is compact.

Pf. Given $\varepsilon > 0$, let

$$K_\varepsilon(x, y) = \begin{cases} K(x, y), & |x-y| > \varepsilon, \\ \frac{Q(x, y)}{\varepsilon^\alpha}, & |x-y| \leq \varepsilon \end{cases}$$

Then K_ε is cont and hence

$$K_\varepsilon f(x) = \int_{\bar{\Omega}} K_\varepsilon(x, y) f(y) dy$$

defines an opt op $\bar{L}^2(\Omega) \rightarrow \bar{L}^2(\Omega)$. Let

$$D_\varepsilon(x, y) = K_\varepsilon(x, y) - K_\varepsilon(y) = \begin{cases} 0, & |x-y| > \varepsilon \\ Q(x, y) \left[\frac{1}{|x-y|^\alpha} - \frac{1}{\varepsilon^\alpha} \right], & |x-y| \leq \varepsilon. \end{cases}$$

Now

$$\int_{\bar{\Omega}} \|K(x, y) f(y)\| \leq \left(\sup_{\bar{\Omega} \times \bar{\Omega}} |Q| \right) \int_{\bar{\Omega}} |x-y|^{-\alpha} dy \leq C_{\Omega, \alpha} \sup_{\bar{\Omega} \times \bar{\Omega}} |Q|$$

$$\int_{\bar{\Omega}} \|K(x, y) f(x)\| \leq \left(\sup_{\bar{\Omega} \times \bar{\Omega}} |Q| \right) \int_{\bar{\Omega}} |x-y|^{-\alpha} dx \leq C_{\Omega, \alpha} \sup_{\bar{\Omega} \times \bar{\Omega}} |Q|$$

So $K: L^2(\Omega) \rightarrow L^2(\Omega)$ is bnd. Now $\forall x \in \Omega$:

$$\begin{aligned} \int_{\Omega} |D_{\varepsilon}(x, y)| dx &\leq \|Q\|_{\infty} \int_{|x-y| < \varepsilon} \frac{1}{|x-y|^{\alpha}} - \frac{1}{\varepsilon^{\alpha}} dy \\ &\leq \|Q\|_{\infty} \left(\int_{\varepsilon}^{\varepsilon} r^{n-1-\alpha} dr + C\varepsilon^{n-\alpha} \right) \quad \text{vol } \{|x-y| \leq \varepsilon\} \sim \varepsilon^n \\ &= \|Q\|_{\infty} \left(\int_0^{\varepsilon} r^{n-1-\alpha} dr + C\varepsilon^{n-\alpha} \right) = O(\varepsilon^{n-\alpha}) \rightarrow 0 \end{aligned}$$

Similarly

$$\|D_{\varepsilon}(x, y)\| \leq C\varepsilon^{n-\alpha}$$

Hence $D_{\varepsilon} = K - K_{\varepsilon} \xrightarrow{L^2(\Omega)} L^2(\Omega)$ with norm $\sim C\varepsilon^{n-\alpha}$ and

hence K is cph as a norm limit of cph ops. \square

So
 FACTS OF LIFE I
 [image of a linear op is not generally a closed subspace]

Under some assumptions this will hold. To this end we define

3.1.1 Def: i) Let $K \in \mathcal{K}(H_1, H_2)$. Then an eqn

$Kx = y, y \in H_2$, is a Fredholm equation of 1st kind

ii) if $H = H_1 = H_2$ and $K \in \mathcal{K}(H_1, H_2)$, an equation

$$(\mathbb{I}_H - K)x = y, y \in H,$$

is a Fredholm equation of 2nd kind.

Like with Volterra eqns, Fredholm eqns of 2nd kind are easier to analyse than 1st kind. This we will now do -

3.2. Fredholm's alternative

In this section, $K: H \rightarrow H$ is ^{a fixed} op on a Hilbert space.

Lemma 3.2.1 $\ker(\mathbb{I} - K)$ is finite dimensional.

Pf. Assume $\dim \ker(\mathbb{I} - K) = +\infty$. Then \exists infinite ON-basis $u_k \in \ker(\mathbb{I} - K), k=1, 2, \dots$ i.e.

\uparrow use Gram-Schmidt

III FREDHOLM-VÄRTÄJÖT JA KOMPAKTIEN

OPERAATTORIEN SPECTRY \leftarrow UPS: again in Finnish.

3.1. Definitions

Let $A \in \mathcal{K}(H_1, H_2), H_1, H_2$ Hilbert. We define

$$\ker A = \{x \in H_1; Ax = 0\}$$

$$\text{im } A = \{y \in H_2; \exists x \in H_1 \text{ s.t. } Ax = y\}$$

"image or range of A"

Note: A cond $\Rightarrow \ker A$ is always a closed subspace.

im A might not be: let $A: \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$,

$$A: (a_n)_{n=1}^{\infty} \mapsto (\bar{e}_n a_n)_{n=1}^{\infty}$$

Then $\|A\| = 1/e$ and assume $\tilde{H} := \text{im } A$ is closed.

If $x = (x_n)_{n=1}^{\infty} \in \tilde{H}^{\perp}$, then

$$0 = \langle (x_n), (\bar{e}_n a_n) \rangle = \sum x_n \bar{a}_n e^{-n} \quad \forall (a_n) \in \ell^2(\mathbb{C})$$

Hence (choose $a_n = \delta_{k,n}$) $x_k \bar{e}^{-k} = 0 \quad \forall k \Leftrightarrow x_k = 0$

$\therefore \tilde{H}^{\perp} = \{0\} \Leftrightarrow \tilde{H} = \ell^2(\mathbb{C})$. This cannot

^{2-we assume \tilde{H} closed}

be true: let $b_n = 1/n$, then $(b_n) \in \ell^2(\mathbb{C})$,

and $A(e_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty} \Leftrightarrow a_n = e^{-n} b_n = e^{-n}/n$.

But $(a_n)_{n=1}^{\infty} \notin \ell^2(\mathbb{C})$.

$K u_k \Rightarrow K u_{k_j} \xrightarrow{H} K u \Rightarrow u_{k_j} \xrightarrow{K u}$
 so $(1/k)$ gives $K u = u$. Hence $u \in \ker(I-K)$

and thus $0 = \langle u_{k_j}, u \rangle = 0, j=1, 2, \dots$

But $u_{k_j} \xrightarrow{w} u \Rightarrow$

$$0 = \lim_j \langle u_{k_j}, u \rangle = \|u\|^2,$$

but since $\|u_{k_j}\| \rightarrow 1, u_{k_j} \xrightarrow{H} u \Rightarrow \|u\| = 1$.

Hence (I) holds.

Assume $\|u_k - v\| \rightarrow 0, v_k = u_k - K u_k$
 and we can take $u_k \in \ker(I-K)^\perp$. Then

$$\|v_k - v\| = \|(u_k - u_e) - \underbrace{K(u_k - u_e)}_{\in \ker(I-K)^\perp} \| \geq \gamma \|u_k - u_e\|$$

$\Rightarrow (u_k)$ is a Cauchy seq. in $H \Rightarrow \exists u = \lim u_k$. But
 Then

$$(I-K)u = \lim (u_k - K u_k) = \lim v_k = v$$

$\Rightarrow v \in \text{im}(I-K) = \{0\}$.

Before the next result we need the following general fact:

Prop. 3.2.3. If $A \in \mathcal{L}(H_1, H_2), H_1, H_2$ Hilbert then
 $\overline{\text{im}(A)} = \ker(A^*)^\perp$.

$\|u_k\| = 1, \langle u_k, u_e \rangle = 0$ when $k \neq l$.
 Now (we've seen this before) if $k \neq l$,

$$\|u_k - u_l\|^2 = \langle u_k - u_l, u_k - u_l \rangle = \underbrace{\|u_k\|^2 + \|u_l\|^2}_{=2} - \underbrace{2\langle u_k, u_l \rangle}_{=0} = 2$$

$\Rightarrow (u_k)_k$ is bnd seq; Also

$$(I-K)u_k = 0 \Leftrightarrow K u_k = u_k. \text{ Hence}$$

$$\|K u_k - K u_l\|^2 = \|u_k - u_l\|^2$$

and hence \exists a convergent subsequence of $(K u_n)_n$. \uparrow \square

Lemma 3.2.2. $\text{im}(I-K)$ is closed.

Pf. Let's first prove that $\exists \gamma > 0$ s.t.

$$(I) \forall u \in \ker(I-K)^\perp : \|u - K u\| \geq \gamma \|u\|.$$

Assume (I) does not hold for any $\gamma > 0$. Then $\forall k \exists$

$$u_k \in H, \|u_k\| = 1,$$

$$(1/k) \quad \|u_k - K u_k\| < 1/k.$$

Hence $u_k - K u_k \rightarrow 0$. Since (u_k) bnd, \exists weakly conv. subsequence (u_{k_j}) ; let $u = w\text{-lim } u_{k_j}$.

But then (I) fails by moving to u in subseq. case (mas)

Pf. It is enough to show that

$$\langle Ax, y \rangle = 0 \quad \forall x \in H_1, y \in \ker(A^*)$$

But $A^*y = 0 \Rightarrow$

$$0 = \langle x, A^*y \rangle = \langle Ax, y \rangle \quad \forall x \in H_1 \quad \square$$

Prop. 3.2.4. If $K \in \mathcal{K}(H_0)$, then

$$\text{im}(\mathbb{I} - K) = \ker(\mathbb{I} - K^*)^\perp$$

Pf. Since $\text{im}(\mathbb{I} - K)$ is closed, this follows from the prev. prop. \square

This is often said as "im($\mathbb{I} - K$) has finite codimension". Namely, since $\text{im}(\mathbb{I} - K)$ is closed we can write

$$H = \ker(\mathbb{I} - K^*) \oplus \ker(\mathbb{I} - K^*)^\perp = \ker(\mathbb{I} - K^*) \oplus \text{im}(\mathbb{I} - K)$$

\uparrow finite dimensional

Prop. 3.2.5 If $K \in \mathcal{K}(H)$, then

$$\ker(\mathbb{I} - K) = \{0\} \Leftrightarrow \text{im}(\mathbb{I} - K) = H.$$

i.e. $\mathbb{I} - K$ is inj $\Leftrightarrow \mathbb{I} - K$ is surj.

Pf. Assume $\ker(\mathbb{I} - K) = \{0\}$, but

$$\text{im}(\mathbb{I} - K) =: H_1 \subsetneq H$$

is a proper closed subspace of H . Let now $H_2 := (\mathbb{I} - K)H_1$.

This is again a closed subspace of $H_1 \subsetneq H$. Assume $H_2 = H_1$.

Then $\forall x \in H \exists y \in H_1$ s.t.

$$\downarrow (\mathbb{I} - K) \text{ inj}$$

$$(\mathbb{I} - K)x = x_1 = (\mathbb{I} - K)y_1 \quad \text{for a unique } y_1 \in H$$

$$\Rightarrow (\mathbb{I} - K)x = y_1. \quad \text{Hence } H_1 = H \quad \nabla.$$

Thus $H \supsetneq H_1 \supsetneq H_2$.

Generally letting $H_k = (\mathbb{I} - K)H_{k-1}$, $k = 2, 3, \dots$

we have a strictly decreasing sequence of closed subspaces

$$H \supsetneq H_1 \supsetneq H_2 \supsetneq \dots \supsetneq H_k \supsetneq H_{k+1} \supsetneq \dots$$

choose $u_k \in H_k$, $\|u_k\| = 1$ \leftarrow We can do this since $\forall k$

and $u_k \perp H_{k+1}$. $H_{k+1} \subsetneq H_k \Rightarrow H_k \neq \{0\}$.

Then $\underbrace{u_k}_{\in H_{k+1}} \in \underbrace{H_{k+1}}_{\in H_{k+1}}$

$$(-1)Ku_k - Ku_k = -(u_k - Ku_k) + (u_k - Ku_k) + (u_k - u_k).$$

Assume $k > l$. Then $u_k \in H_k \subset H_{l+1}$. Hence

$$-(u_k - Ku_k) + (u_k - Ku_k) + u_k \subset H_{l+1}$$

$$u_k \in H_{l+1}, \|u_k\| = 1$$

so $(\cdot) \Rightarrow$

$\|Ku_k - Ku_k\|^2 \geq \|u_k\|^2 = 1$. This is a contradiction since

K cpl. Thus

$$(im) \quad \text{im}(\mathbb{I} - K) = H.$$

Conversely assume (im) holds. Prop. 3.2.4 \Rightarrow

$$\ker(\mathbb{I} - K^*) = \{0\}$$

$$\text{so } \text{im}(\mathbb{I} - K^*) = H$$

$$\ker(\mathbb{I} - K)^\perp \Rightarrow \ker(\mathbb{I} - K) = \{0\}. \quad \square$$

Prop. 3.2.6 $\dim \ker(\mathbb{I}-K) = \dim \ker(\mathbb{I}-K^*)$.

Prf. It is enough to prove

(i) ~~$\dim \ker(\mathbb{I}-K) \geq \dim \ker(\mathbb{I}-K^*)$~~ $\dim \ker(\mathbb{I}-K) \geq \dim \ker(\mathbb{I}-K^*)$

For $\dim(\mathbb{I}-K)^\perp = \ker(\mathbb{I}-K^*)$,

so (i) implies

(ii) $\dim \ker(\mathbb{I}-K) \geq \dim \ker(\mathbb{I}-K^*)$ and the claim follows from this by applying (ii) to K^* .

To prove (ii) we argue by contradiction: assume

(iii) $\dim \ker(\mathbb{I}-K) < \dim \ker(\mathbb{I}-K^*)$.

Let

$A: \ker(\mathbb{I}-K) \rightarrow \ker(\mathbb{I}-K^*)^\perp$

be first, linear and not onto: here we use (iii). Extend

$A: H \rightarrow \ker(\mathbb{I}-K^*)^\perp$ s.t. $A|_{\ker(\mathbb{I}-K)} = 0$.

A is finite dimensional $\Rightarrow K+A$ is spl.

Also

$0 = (\mathbb{I} - (K+A))u \Leftrightarrow u = Ku + Au \Leftrightarrow \underbrace{u - Ku}_{\in \ker(\mathbb{I}-K)} = Au \in \ker(\mathbb{I}-K)^\perp$

$\Rightarrow 0 = u - Ku = Au \Rightarrow \begin{cases} u \in \ker(\mathbb{I}-K) \\ Au = 0 \end{cases} \Rightarrow \begin{cases} u \in \ker(\mathbb{I}-K) \\ Au = 0 \end{cases} \Rightarrow u = 0$.

Hence $\mathbb{I} - (K+A)$ is inv. Then Prop. 3.2.5 \Rightarrow

$\dim(\mathbb{I} - (K+A)) = H$.

This is not true: choose $v \in \ker(\mathbb{I}-K)^\perp, v \notin \ker(\mathbb{I}-K)$.

Then \uparrow

$\underbrace{u - Ku - Au}_{\in \ker(\mathbb{I}-K)} = v \in \ker(\mathbb{I}-K) \xrightarrow{\text{Proj.}} -Au = v \notin \ker(\mathbb{I}-K) \uparrow \square$

3.2.7 Froehlich's alternative we can summarize the above as follows: Let $K \in \mathcal{K}(H, H)$;

(A) if $\mathbb{I}-K$ is inv, then it is onto, and conversely \uparrow $\mathbb{I}-K$ is onto, then $\mathbb{I}-K$ is inv.

(B) if $\mathbb{I}-K$ is not inv, let $\ell < \infty$ be $\ell = \dim \ker(\mathbb{I}-K)$; Then $\ell = \dim \ker(\mathbb{I}-K^*) = \dim \ker(\mathbb{I}-K)^\perp$ and hence

eqn $(\mathbb{I}-K)u = v \perp \ker(\mathbb{I}-K)$

has a solution u if and only if $v \perp \ker(\mathbb{I}-K)$ i.e. \exists on vectors $\beta_1, \dots, \beta_\ell \in \ker(\mathbb{I}-K)^\perp$ s.t.

$\langle v, \beta_j \rangle = 0, \forall j=1, \dots, \ell$.

Also u is unique only modulo a subspace of dimension ℓ .

3.3. Spectrum of spl operators

Recall that if A is a V -matrix, $\lambda \in \mathbb{C}$ is an eigenvalue

\uparrow $A - \lambda I$ is not invertible. For a general $n \times n$ -matrix, eigenvalues are the (complex) roots of the polynomial (of degree n)

$0 = \det(A - \lambda I)$, hence the an n eigenvalues (counting multiplicities)

For a general $A \in \mathcal{L}(H, H)$, H Hilbert, we define:

3.3.1 Def. (i) The resolvent set $\rho(A)$ of A is $\rho(A) = \{ \lambda \in \mathbb{C} ; A - \lambda I \text{ is a bijection } H \rightarrow H \}$

(ii) The spectrum of A , $\sigma(A)$ is

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Rem. Note that Open Mappings Thm $\Rightarrow (A - \lambda I)^{-1}$ is End if $\lambda \in \rho(A)$.

Hence we could equally well define

$$\rho(A) = \{ \lambda \in \mathbb{C} ; A - \lambda I \text{ is invertible} \}.$$

Ex. 3.3.2. Let $H = \mathbb{L}^2([0, \infty])$, $\alpha > 0$ and let

$$A : f \mapsto x^\alpha f;$$

then $\|A\| \leq \alpha$; let $\lambda \in \mathbb{C}$;

$$(A - \lambda)f = (x - \lambda)f = 0 \Leftrightarrow f = 0 \text{ a.e.}$$

i.e. $A - \lambda$ is always inj. But given $g \in \mathbb{L}^2([0, \infty])$

$$(A - \lambda)f = g \Leftrightarrow f(x) = \frac{g}{x - \lambda} \in \mathbb{L}^2 \text{ iff } |x| > \alpha.$$

Hence

$$\rho(A) = \{ \lambda \in \mathbb{C} ; |\lambda| > \alpha \}$$

$$\sigma(A) = \{ \lambda \in \mathbb{C} ; |\lambda| \leq \alpha \}.$$

Note that the situation is very different from the finite dimensional

case: For matrices $\lambda - A$ inj. $\Leftrightarrow \lambda - A$ surj.; hence

$\lambda - A$ not invertible $\Leftrightarrow \exists x \neq 0$ (eigenvector) s.t.

$$Ax = \lambda x.$$

This is not the case for the operator of Ex. 3.3.2: injectivity is always valid but the surjectivity fails!

To be able to be more precise, let:

Def. 3.3.3. $\lambda \in \sigma(A)$ is an eigenvalue of A if $\exists x \in H, x \neq 0$ s.t.

$$Ax - \lambda x = 0.$$

We call $x \neq 0$ satisfying (i) an eigenvector (corresponding to eigenvalue λ). Let

$$\sigma_p(A) = \{ \lambda \in \mathbb{C} ; \lambda \text{ eigenvalue} \}.$$

Set $\sigma_p(A)$ is the point spectrum of A .

So for the ^{operator in} example 3.3.2,

$$\sigma(A) = [0, \alpha], \quad \sigma_p(A) = \emptyset.$$

For compact A one might think that $\sigma(A)$ behaves more like the spectrum of a matrix (since they are norm limits of ops with finite range). This is true for certain extend:

Thm. 3.3.4. Assume H is an infinite dimensional Hilbert space. Then if $K \in \mathcal{K}(H, H)$,

$$(i) 0 \in \sigma(K)$$

$$(ii) \sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$$

(iii) $\delta(K) \setminus \{0\}$ is either finite or a sequence converging to 0.

Pf. (i) Assume $0 \notin \delta(K)$. Then $\exists K' \in \mathcal{L}(H, H)$ and thus $\Pi = K K^{-1}$ is spd. Since $\dim H = \infty$ this is a contradiction.

(ii) Let $\lambda \in \delta(K)$. If $\ker(\lambda I - K) = \{0\}$, Fredholm alt. $\Rightarrow \text{im}(\lambda I - K) = H \Rightarrow \lambda \notin \delta(K)$.

(iii) Assume now $(\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct elements of $\delta(K) \setminus \{0\}$. We will prove the following:

(lim) $\lambda = \lim_{k \rightarrow \infty} \lambda_k \Rightarrow \lambda = 0$ if $\delta(K) \setminus \{0\}$ is unbounded, this will imply that $\delta(K) \setminus \{0\}$ consists of a sequence tending to 0: first, if $\lambda > \|K\|$, then

$$\lambda I - K = \lambda(I - \frac{1}{\lambda}K)$$

is invertible by the Neumann-series. Since $\delta(K) \subset [-\|K\|, \|K\|]$, \in set of eigenvalues outside $\{0\}$ is inf; it has accumulation pts. By (lim) 0 is the only possible accumulation pt, proving the claim. So let's prove (lim): Given $\lambda_k \in \delta(K) \setminus \{0\} = \phi_p(K) \setminus \{0\}$, $\exists w_k \neq 0$ s.t.

$$K w_k = \lambda_k w_k$$

Let $H_k = \text{span}\{w_1, \dots, w_k\}$.

Now we need a

Lemma 3.3.5. If $(\lambda_j)_{j=1}^{\infty}$ are distinct, then $(w_k)_{k=1}^{\infty}$ are linearly independent.

We will prove this in a moment; let's first prove the Thm. using this. Lemma 3.3.5 \Rightarrow

$$H_k \supsetneq H_{k+1}$$

Also

$$(K - \lambda_k \Pi) H_k \subseteq H_{k-1}$$

since $(K - \lambda_k \Pi) w_i = \begin{cases} 0, & i=k \\ K w_i - \lambda_k w_i \in H_{k-1}, & i < k \end{cases}$

Choose now

$$u_k \in H_k, u_k \in H_{k-1}^{\perp}, \|u_k\| = 1.$$

If $k > l$,

$$H_{l-1} \supsetneq H_l \subseteq H_{k-1} \supsetneq H_k,$$

and

$$\begin{aligned} \|\frac{1}{\lambda_k} K u_k - \frac{1}{\lambda_l} K u_l\| &= \|\underbrace{\frac{K u_k - \lambda_k u_k}{\lambda_k}}_{\in H_{k-1}} - \underbrace{\frac{K u_l - \lambda_l u_l}{\lambda_l}}_{\in H_{l-1}}\| \\ &\geq \|u_k\| = 1 \end{aligned}$$

If now $\lambda_k \rightarrow \lambda \neq 0$ this contradicts the compactness of K . \square

Let's prove lemma 3.3.5:

Pf of Lemma 3.3.5 : Assume $\sum_{i=1}^n \alpha_i w_i = 0, \alpha_i \in \mathbb{C}$.

where $K_{ij} = \lambda_i \lambda_j$, $\omega_k \neq 0$ & λ_k distinct.
Then $\forall \ell$

$$0 = K^\ell \left(\sum_{i=1}^n a_i \omega_i \right) = \sum_{i=1}^n \lambda_k^\ell a_i \omega_i$$

and hence taking $\ell=0, \dots, n-1$ we get

$$\begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \omega_1 \\ \vdots \\ a_n \omega_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

But $= \Lambda^{n \times n}$

$$\det \Lambda^{n \times n} = \prod_{i < j} (\lambda_i - \lambda_j) \quad (\text{Vandermonde})$$

$$\neq 0 \quad \text{in } a_1 \omega_1 = \dots = a_n \omega_n = 0 \Rightarrow a_k = 0 \forall k. \quad \square$$

We can prove little more if we assume that K is self-adjoint.

Prop. 3.3.6 If $K = K^*$ then all eigenvalues of K are real, and eigenvectors corresponding to disjoint eigenvalues are orthogonal.

Ex. 3.3.6 like in the linear algebra course:
Assume $Kx = \lambda x$, $x \neq 0$. Then $K = K^* \Rightarrow$

$$\lambda \langle x, x \rangle = \langle Kx, x \rangle = \langle x, Kx \rangle = \overline{\lambda} \langle x, x \rangle$$

$x \neq 0 \Rightarrow \lambda = \overline{\lambda}$ i.e. λ real.

Also, if $Kx = \lambda x$, $Ky = \mu y$, $x, y \neq 0$, $\mu \neq \lambda$, then $K = K^* \Rightarrow$

$$\lambda \langle x, y \rangle = \langle Kx, y \rangle = \langle x, Ky \rangle = \overline{\mu} \langle x, y \rangle = \overline{\mu} \langle x, y \rangle$$

$$\Leftrightarrow (\lambda - \overline{\mu}) \langle x, y \rangle = 0, \text{ But } \lambda \neq \overline{\mu} \Rightarrow \langle x, y \rangle = 0. \quad \square$$

3.4. Dirichlet eigenvalues

As an application of all the above, we study the following important question.

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. We want to understand what kind of "vibrations" Ω can have. So, let's start with the wave eqn in Ω :

$$(\square) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega;$$

here u is the "vertical displacement" of Ω from equilibrium state $u=0$. (be "small" i.e. scale units s.t. wave-speed $\equiv 1$ in Ω .)

By an "eigenmode" or "vibration" we mean the following:

(1) $\frac{\partial u}{\partial t} = 0$ is clamped at $\partial\Omega$ i.e. $u|_{\partial\Omega} = 0 \quad \forall t$.

(2) u has freq. $\omega > 0$ i.e.

$$u(t, x) = e^{i\omega t} \psi(x).$$

Now plugging (2) to (1) gives

$$-\omega^2 e^{i\omega t} (U(x)) + e^{i\omega t} \Delta U = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{Laplace-op}$$

$$\Leftrightarrow (-\Delta - \omega^2)U = 0.$$

Hence (in some vague sense) U must be an eigenfunction of $-\Delta$ with eigenvalue ω^2 .

$$\text{Also } \textcircled{1} \Rightarrow U|_{\partial\Omega} = 0.$$

Hence we want to know what we can mathematically say about those $\omega > 0$ that have $U \neq 0$ satisfying

$$\begin{cases} \Delta U = -\omega^2 U \\ U|_{\partial\Omega} = 0. \end{cases}$$

However, let's start with the following:

(2.) Weak solutions for the ^{Poisson} Dirichlet problem

We want to solve

$$(P) \quad \begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

This is a classical, well understood problem that however has closed form solutions only in (generally) symmetric special domains Ω . We want to use Hilbert-space methods, and the 1st grand question is then how to formulate (P) in some Hilbert space?