Integral equations

HW 8

Again only five exercises.

1. Let $\Phi_k(x) = e^{ik|x|}/4\pi|x|$ and assume that $\varphi \in C_0(\mathbb{R}^3)$. Prove that
$$u(x) = \int \Phi_k(x-y)\,\varphi(y)\,dy$$
satisfies Sommerfeld's radiation condition.

2. Assume that $\Omega \subset \mathbb{R}^3$ is a bounded C^2–domain with a connected complement. Assume that $u \in C^2(\mathbb{R}^3 \setminus \Omega)$ satisfies Sommerfeld's radiation condition and solves $\Delta u + k^2 u = 0$. Show that if $x \in \mathbb{R}^3 \setminus \Omega$, then
$$u(x) = \int_{\partial\Omega} u(y)\frac{\Phi_k(x-y)}{\partial\nu(y)} - \frac{\partial u(y)}{\partial\nu(y)}\Phi_k(x-y)\,dS(y).$$
Here ν is the exterior unit normal to Ω.

3. With otherwise the same assumptions as in Exercise 2, show that if $x \in \Omega$, then
$$0 = \int_{\partial\Omega} u(y)\frac{\Phi_k(x-y)}{\partial\nu(y)} - \frac{\partial u(y)}{\partial\nu(y)}\Phi_k(x-y)\,dS(y).$$

4. Consider the scattered field as defined in the lectures:
$$u_s(x) = -k^2\int_{\mathbb{R}^3}\Phi_k(x-y)\,m(y)u(y)\,dy,$$
where $u \in C_0^2(\mathbb{R}^3)$. Show that u has the asymptotics
$$u_s(x) = \frac{e^{ik|x|}}{|x]}u_\infty(x/|x|) + \mathcal{O}(|x|^{-2}),\ |x| \to \infty.$$
The function u_∞ is called the *far field pattern* of u_s. Give an explicit formula for it.

5. Let now $u_i(x) = e^{ik\langle x,d\rangle}$, where $|d| = 1$. Let $u_\infty(x/|x|, d)$ be the far field pattern of the corresponding scattered field. Show that it satisfies the following *reciprocity relation*
$$u_\infty(x/|x|, d) = u_\infty(-d, -x/|x|).$$