Integral equations

HW 7

Again only five exercises.

1. Show that the set of Dirichlet eigenvalues of Δ on $\Omega \subset \mathbb{R}^d$ is invariant under rotations, reflections and translations of Ω.

2. Given $\lambda > 0$ and $\Omega \subset \mathbb{R}^d$, let $\lambda\Omega = \{\lambda x;\, x \in \Omega\}$. What can you say about the Dirichlet–eigenvalues of $\lambda\Omega$?

   For the next two exercises fix a bounded domain $\Omega \subset \mathbb{R}^d$, let

   $$C^2_\partial(\Omega) = \{u \in C^2(\Omega) \cap C(\overline{\Omega});\, u|_{\partial\Omega} = 0\}$$

   and define

   $$\lambda_1 = \inf_{w \in C^2_\partial(\Omega)} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}}.$$

3. Assume $u \in C^2_\partial(\Omega)$ is such that

   $$\lambda_1 = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

   i.e we attain the minimum at u. Prove that λ_1 is a Dirichlet eigenvalue of $-\Delta$ on Ω with eigenvalue u. **Hint**: Given any $v \in C^2_\partial(\Omega)$ study the function

   $$f(\varepsilon) = \frac{\|\nabla(u + \varepsilon v)\|^2_{L^2(\Omega)}}{\|u + \varepsilon v\|^2_{L^2(\Omega)}},$$

   at origin.

4. Prove that the $\lambda_1 \leq \lambda$ for all Dirichlet eigenvalues λ of $-\Delta$ on Ω.

5. Define

   $$K(s,t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\{\cos(k+1)s \sin kt - \sin(k+1)s \cos kt\}, \;\; s,\, t \in \mathbb{R}.$$

   Prove that the integral operator with kernel $K(s,t)$ on interval $[0, 2\pi]$ has no eigenvalues.