Integral equations

HW 6

1. Define
$$x_+^a = \begin{cases} x^a, \ x > 0 \\ 0, \ x \le 0. \end{cases}$$
Determine those values $a \in \mathbb{R}$ for which x_+^a has a weak derivative in the sense that we defined in the lectures.

   For the next three exercises we assume that H is a **real** Hilbert space. Especially the inner product $\langle \cdot, \cdot \rangle$ is an $\mathbb{R}$–bilinear map on $H \times H$.

2. Assume that $B : H \times H \to \mathbb{R}$ is a real bilinear map for which there exists constants M, $m > 0$ such that
$$|B(u, v))| \le M \|u\| \, \|v\|, \quad u, \, v \in H,$$
and
$$m\|u\|^2 \le B(u, u), \quad u \in H.$$
Prove that there is a unique bounded linear operator $A : H \to H$ such that
$$B(u, v) = \langle Au, v \rangle, \quad u, \, v \in H.$$

3. Prove that the operator A constructed above is a bijection.

4. Prove now the *Lax-Milgram Theorem*: If B is as above and $\lambda : H \to \mathbb{R}$ is a bounded linear functional, then there exists a unique element $u \in H$ such that for all $v \in H$ we have
$$B(u, v) = \lambda(v).$$

Let now $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the linear partial differential operator
$$L = -\Delta + \sum_{k=1}^{n} b^k(x) \frac{\partial}{\partial x_k} + c(x)$$
where the real valued functions b_k and c are continuous in $\overline{\Omega}$.

5. Define the bilinear form

$$B(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle + \sum_1^n b^k \frac{\partial u}{\partial x_k} v + cuv\, dx$$

on $H_0^1(\Omega)\times H_0^1(\Omega)$. Prove that B satisfies the so-called energy estimates: there exists positive constants M, m and C such that

$$|B(u,v)| \le M\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$$

and

$$m\|u\|_{H_0^1(\Omega)}^2 \le B(u,u) + C\|u\|_{L^2(\Omega)}^2$$

for all u, $v \in H_0^1(\Omega)$.

6. Apply the previous exercise to study the weak solvability on $H_0^1(\Omega)$ of the boundary value problem

$$Lu + \mu u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

for a large enough constant μ.