Integral equations

HW 5

Again there are only five exercises.

1. Let's define the shift operator $S : \ell^2 \to \ell^2$ by
$$Sx(n) = \begin{cases} 0, \; n = 0 \\ x(n-1), \; n \geq 1. \end{cases}$$
Here $x = (x(n))_{n=0}^{\infty}$. Also let $M : \ell^2 \to \ell^2$ be defined by
$$Mx(n) = (n+1)^{-1}x(n),$$
Show that $T = MS$ is a compact operator that has no eigenvalues. Hence the spectrum consists only of $\{0\}$.

2. Let
$$K(s,t) = \begin{cases} (1-s)t, \; 0 \leq t \leq s, \\ (1-t)s, \; s \leq t \leq 1. \end{cases}$$
prove that
$$\mathcal{K}f(s) = \int_0^1 K(s,t)f(t)\,dt$$
defines a compact operator $L^2[0,1] \to L^2[0,1]$. Determine its eigenvalues and eigenspaces.

3. Prove that
$$Tf(s) = \frac{1}{s}\int_0^s f(t)\,dt$$
defines a bounded operator $L^2(0,\infty) \to L^2(0,\infty)$. Is T compact? **Hint**: Hardy's inequality :)

4. Give an example of A, $B \in \mathcal{L}(H)$ such that $AB = I$ but $BA \neq I$.

5. Using the notation in the previous exercise, assume that B is compact. Prove that
$$A(I-B) = I \Leftrightarrow (I-B)A = I.$$