Integral equations

HW 4

Note that there are only five exercises this time.

1. Give an example of a bounded linear operator between Hilbert spaces whose image is not a closed subspace.

2. Assume $K \in \mathcal{L}(H_1, H_2)$ and that for some positive integer n_0 we know that K^{n_0} is compact. What can you say about $\ker(I - K)$?

3. Assume A, $B \in \mathcal{L}(H, H)$ commute, i.e. $AB = BA$. If AB is invertible, what can you say about the invertibility of A and B?

4. Consider the integral operator

$$\mathcal{K}u(x) = \int_a^b K(x, y)u(y)\, dy, \quad x \in (a, b).$$

Assume that $K \in L^2([a, b])$. Prove that $\mathcal{K}$ is compact $L^2([a, b]) \to L^2([a, b])$.

5. Prove that a compact operator $K : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is a norm limit of finite dimensional operators. **Hint**: Let Q_n be the orthogonal projection to $\operatorname{span}(e_1, \ldots, e_n)$, where (e_i) is the standard orthonormal basis of $\ell^2(\mathbb{C})$. Let $K_n = Q_n K$ and prove that $\|K - K_n\| \to 0$ by considering a suitable finite covering of the compact set $\overline{K(B)}$ where B is the closed unit ball of $\ell^2(\mathbb{C})$.