Integral equations

HW 3

1. Let $(X_i, \|\cdot\|_i)$ be normed spaces, $i = 1, 2, 3$. Show that for the norm of a linear operator $A : X_1 \to X_2$ we have
$$\|A\| = \sup_{\|x\|_1 \le 1} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 = 1} \frac{\|Ax\|_2}{\|x\|_1}.$$
Let also $B : X_2 \to X_3$ be linear. Prove that
$$\|BA\| \le \|B\|\|A\|.$$

2. Consider the integral equation
$$f(x) + \frac{1}{20}\int_0^1 e^{-|xy|^2} \sin(x^2 + y^2) f(y)\, dy = \sin x.$$
Prove that this has a unique solution in $L^2([0,1])$, and that in fact this solution is also continuous.

3. Let H be a Hilbert space, and $A : H \to H$ be a linear map for which $\|A^{n_0}\| < 1$ for some positive integer n_0. Prove that $I - A$ is invertible and determine its inverse.

4. Assume $K : H_1 \to H_2$ is a compact operator between Hilbert spaces. Given bounded linear maps $A : H \to H_1$ and $B : H_2 \to H$, where H is again Hilbert, prove that KA and BK are compact. Also, prove that the sum $K_1 + K_2$ of two compact operators K_1, $K_2 : H_1 \to H_2$ is compact.

5. Assume $K_n : H_1 \to H_2$, $n = 1, 2, \ldots$, are compact and that $\|A - K_n\| \to 0$ as $n \to \infty$. Here $A \in \mathcal{L}(H_1, H_2)$. Prove that A is compact.

6. Assume (a_n) is a sequence of complex numbers converging to zero. Consider the linear map
$$A : \ell^2 \to \ell^2, \quad (x_n) \mapsto (a_n x_n).$$
Prove that K is compact. **Hint:** use the previous exercise with suitable operators K_n having finite dimensional image spaces.