Integral equations

HW 2

1. Solve the Volterra equation of the first kind
$$\int_1^s (s+t)\phi(t)\,dt = s^3 - 1.$$

2. Consider a Volterra equation of the first kind
$$\int_a^s K(s,t)\phi(t)\,dt = f(s) \tag{0.1}$$
where K and f are continuous. Assume $K(s,s) = 0$ for all $s \in [a,b]$ and that K has continuous partial derivatives with respect to s up to second order. Formulate and prove a solvability result for (0.1).

3. Consider the example from mechanics in Section 1.6 of lecture notes: find the solution in the case when $f(x) = T$, i.e when a particle is released from height $x > 0$, it always takes a constant time $T > 0$ to travel along the curve $y = F(x)$ to zero height. Find the equation of F, or at least a series approximation to it.

4. Consider a nonlinear Volterra equation of second kind.
$$\phi(s) + \int_0^s K(s,t,\phi(t))\,dt = f(s). \tag{0.2}$$
Assmue the following: the function $K(x,y,z)$ is continuous in the set D defined by
$$|x|,\ |y| \le a, \quad |z| \le b,$$
and that K is uniformly Lipschitz-continuous in z–variable,
$$|K(x,y,z_1) - K(x,y,z_2)| \le K|z_1 - z_2|, \quad (x,y,z_i) \in D.$$
Also, assume that $f \in C([-a,a])$, $f(0) = 0$ and that f satisfies a Lipschitz-condition
$$|f(x_1) - f(x_2)| \le k|x_1 - x_2|, \quad |x_i| \le a.$$
Let
$$M = \sup_D |K|.$$

Show that the iteration

$$\phi_0(s) = f(s), \quad \phi_n(s) = f(s) - \int_0^s K(s, t, \phi_{n-1}(t))\, dt$$

converges in the set

$$|s| \leq a', \quad a' = \min\{a, \frac{b}{k + M}\}.$$

and that the limit is a solution of (0.2) on interval $[-a', a']$.

5. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\| \cdot \|$ the induced norm. Prove that an inner product satisfies the so called parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x,\, y \in X.$$

6. Consider the space $C([a, b])$. Show that the sup–norm,

$$\|f\|_{\text{sup}} = \sup_{[a,b]} |f|$$

is not determined by any inner product.