Stochastic analysis, autumn 2011, Exercises-9, 15.11.2011

1. (Martingale characterization) Let $M_t(\omega)$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $E(|M_t|) < \infty\ \forall t$. Then (X_t) is an $\mathbb{F}$-martingale if and only if $\forall$ $\mathbb{F}$-stopping times $\tau(\omega)$ which can take at most two finite values

$$E(M_\tau) = E(M_0)$$

Hint: the necessity is just Doob optional sampling theorem, for sufficency for $s \le t$ and $A \in \mathcal{F}_s$ define $\tau(\omega) = t\mathbf{1}_A + s\mathbf{1}_{A^c}$ and show that it is a stopping time. Then use the hypothesis to show that $E\big((M_t - M_s)\mathbf{1}_A\big) = 0$.

Solution

$$\{\tau \le r\} = \begin{cases} \Omega & \text{when } t \le r \\ A^c & \text{when } s \le r < t \\ \emptyset \text{ when } 0 \le s < r \end{cases}$$

and we see that in all cases $\{\tau \le r\} \in \mathcal{F}_r$.

$$0 = E\big(M_\tau - M_s\big) = E\big(M_t\mathbf{1}_A + M_s(1 - 1_A) - M_s\big)$$
$$= E\big((M_t - M_s)\mathbf{1}_A\big)$$

2. (Lenglart's inequality)

Let $X_t(\omega) \ge 0$ and $A_t(\omega) \ge 0$ continuous processes adapted with respect to $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $X_0 = 0$, and we assume that A_t is non-decreasing such that for all **bounded** stopping times $\tau(\omega)$

$$E(X_\tau) \le E(A_\tau)$$

We introduce the running maximum $X_t^*(\omega) = \max_{0 \le s \le t} X_s(\omega)$

Prove the following inequalities for **all** $\mathbb{F}$-stopping times τ, (also unbounded): $\forall \varepsilon, \delta > 0$

a) $$P(X_\tau^* > \varepsilon) \le \frac{E(A_\tau)}{\varepsilon}$$

b) $$P(X_\tau^* > \varepsilon, A_\tau \le \delta) \le \frac{E(A_\tau \wedge \delta)}{\varepsilon}$$

c) $$P(X_\tau^* > \varepsilon) \le \frac{E(\delta \wedge A_\tau)}{\varepsilon} + P(A_\tau > \delta)$$

Hint: show it first for bounded stopping times, then use monotone convergence.

Solution Let τ be a $\mathbb{F}$-stopping time. Define also $\sigma = \inf\{s : X_s > \varepsilon\}$

Note that

$$X_\sigma \mathbf{1}(\sigma \le t \wedge \tau) = X_{\tau\wedge\sigma\wedge t}\mathbf{1}(X^*_{\tau\wedge\sigma\wedge t} > \varepsilon) =$$
$$\ge \varepsilon \mathbf{1}(X^*_{\tau\wedge\sigma\wedge t} > \varepsilon)$$

Taking expectation and using the assumption

$$\varepsilon P\big(X^*_{\tau\wedge\sigma\wedge t} > \varepsilon\big) \le E\bigg(X_{\tau\wedge\sigma\wedge t}\mathbf{1}(X^*_{\tau\wedge\sigma\wedge t} > \varepsilon)\bigg) \le E\big(X_{\tau\wedge\sigma\wedge t}\big) \le E\big(A_{\tau\wedge\sigma\wedge t}\big)$$

and as $t \uparrow \infty$ by monotone convergence on the left and right hand side

$$\varepsilon P\big(X^*_\tau > \varepsilon\big) = \varepsilon P\big(X^*_{\tau\wedge\sigma} > \varepsilon\big)$$
$$\le E\big(A_{\tau\wedge\sigma}\big) \le E\big(A_\tau\big)$$

Let $\rho = \inf\{s : A_s > \delta\}$.

$$P(X^*_\tau > \varepsilon, A_\tau \le \delta) = P(X^*_{\tau\wedge\rho} > \varepsilon)$$
$$\le \frac{1}{\varepsilon}E(A_{\tau\wedge\rho}) \le \frac{1}{\varepsilon}E(A_\tau \wedge \delta)$$

and since

$$P(X^*_\tau > \varepsilon) \le P(X^*_\tau > \varepsilon, A_\tau \le \delta) + P(A_\tau > \delta)$$

the last equation follows.

3. Let M_t a continuous $\mathbb{F}$-local martingale. The $\mathbb{F}$-predictable variation $\langle M\rangle_t$ is the non-decreasing process with $\langle M\rangle_0 = 0$ such that

$$M_t^2 - \langle M\rangle_t$$

is a $\mathbb{F}$-local martingale.

Show that for any $\mathbb{F}$-stopping time τ

$$P\bigg(\max_{0\le s\le t}|M_s(\omega)| > \varepsilon\bigg) \le \frac{E(\delta\wedge\langle M\rangle_\tau)}{\varepsilon^2} + P(\langle M\rangle_\tau > \delta)$$

Solution Let $\tau'_n \uparrow \infty$ be a localizing sequence for the martingale (M_t). Note that also $(\tau'_n \wedge n)$ is a localizing sequence since if $(M_{t\wedge\tau_n} : t \in \mathbb{R}^+)$ is a true martingale, also the stopped process $(M_{t\wedge\tau_n\wedge n} : t \in \mathbb{R}^+)$ is a true martingale. This means that we can always choose a localizing sequence of bounded stopping times.

Let also $\tau''_n \uparrow$ be a localizing sequence for the local martingale $(M_t^2 - \langle M\rangle_t)$. By choosing $\tau_n = \tau'_n \wedge \tau''_n \wedge n$ we obtain a sequence of stopping

So $(M_{t\wedge\tau_n} : t \in \mathbb{R}^+)$ and $(M^2_{t\wedge\tau_n} - \langle M\rangle_{t\wedge\tau_n} : t \in \mathbb{R}^+)$ are true martingales and τ_n is bounded for each n.

By Doob optional stopping theorem, for every stopping time σ

$$E(M^2_{\tau_n\wedge\sigma}) = E(\langle M\rangle_{\tau_n\wedge\sigma})$$

By Lenglart's inequality

$$P\bigg(\sup_{0\le s\le\tau_n\wedge\sigma}|M_s| > \varepsilon\bigg) \le \frac{1}{\varepsilon^2}E_P\bigg(\langle M\rangle_{\tau_n\wedge\sigma}\bigg)$$

and also

$$P\left(\sup_{0\le s\le \tau_n\wedge\sigma}|M_s|>\varepsilon\right)\le \frac{1}{\varepsilon^2}E_P\left(\langle M\rangle_{\tau_n\wedge\sigma}\wedge\delta\right)+P(\langle M\rangle_{\tau_n\wedge\sigma}>\delta)$$

and as $\tau_n\uparrow\infty$ for $n\uparrow\infty$, by monotone convergence we get

$$P\left(\sup_{0\le s\le \sigma}|M_s|>\varepsilon\right)\le \frac{1}{\varepsilon^2}E_P\left(\langle M\rangle_{\sigma}\right)$$
$$P\left(\sup_{0\le s\le \sigma}|M_s|>\varepsilon\right)\le \frac{1}{\varepsilon^2}E_P\left(\langle M\rangle_{\sigma}\wedge\delta\right)+P(\langle M\rangle_{\sigma}>\delta)$$

4. $\{M_t^{(n)}(\omega)\}_{n\in\mathbb{N}}$ a sequence of $\mathbb{F}$-local martingales and τ a $\mathbb{F}$-stopping time. Show that

$$\langle M^{(n)}\rangle_\tau \overset{P}{\to} 0 \quad\Longrightarrow\quad \max_{0\le s\le\tau}|M_s(\omega)|\overset{P}{\to} 0$$

where we use convergence in probability.

5. (A discontinous martingale) Consider a random time $\tau(\omega)\in[0,\infty]$ with distribution function $F(t)=P(\tau\le t)$.

Define the Riemann Stieltjes integrals

$$\Lambda_t=\int_0^t\frac{1}{1-F(s-)}F(ds)=\int_0^t\frac{1}{P(\tau\ge s)}F(ds)=\int_0^t P(\tau\in ds|\tau\ge s),\quad t\ge 0$$

For simplicity you can assume that F is continuous or even absolutely continuous.

Consider counting process $N_t(\omega):=\mathbf{1}(\tau(\omega)\le t)$ which has one jump at size 1 at time τ, And let $\mathbb{F}=(\mathcal{F}_t:t\in\mathbb{R}^+)$. with $\mathcal{F}_t=\sigma(N_s:s\le t)$.

- Show that τ is $\mathbb{F}$-stopping time.
- Show that $M_t:=N_t(\omega)-\Lambda(t\wedge\tau(\omega))$ is a $\mathbb{F}$-martingale.
  Hint: show that

  $$\mathcal{F}_s=\left\{\Omega,\emptyset,\{\tau(\omega)>s\}\text{and }\{\tau(\omega)\in B\},\{\tau\in(s,\infty)\cup B\}:\ B\subset[0,s]\text{ Borel }\right\}$$

  Solution For fixed s, $(a,b]\subseteq[0,s]$, since τ is a stopping time

  $$\{\tau\in(a,b]\}=\{\tau\le b\}\setminus\{\tau\le a\}\in\mathcal{F}_s$$

  by taking countable unions and intersections it follows $\{\tau\in B\}\in\mathcal{F}_s$ for all Borel sets $B\in[0,s]$
  Also $\{\tau>s\}=\{\tau\le s\}^c\in\mathcal{F}_s$. There are no other sets.
  In fact when $B\subseteq[0,s]$ obviously $\{\tau>s\}\cap\{\tau\in B\}=\emptyset\in\mathcal{F}_s$ and $\{\tau\in(s,\infty)\cup B\}\in\mathcal{F}_s$
  For such sets $A\in\mathcal{F}_s$ and $s\le t$ compute $E((M_t-M_s)\mathbf{1}_A)$.
  Solution For $A=\{\tau\in B\}$Âăwith $B\subset[0,s]$ Borel

  $$N_t\mathbf{1}(\tau\in B)=N_s\mathbf{1}(\tau\in B)=\mathbf{1}(\tau\in B)$$
  $$\Lambda_{t\wedge\tau}\mathbf{1}(\tau\in B)=\Lambda_{s\wedge\tau}\mathbf{1}(\tau\in B)=\Lambda_\tau\mathbf{1}(\tau\in B)$$

and

$$0 = E\Bigg((N_t - N_s)\mathbf{1}(\tau \in B)\Bigg) = E\Bigg((\Lambda_{t\wedge\tau} - \Lambda_{s\wedge\tau})\mathbf{1}(\tau \in B)\Bigg)$$

Also

$$E((N_t - N_s)\mathbf{1}(\tau > s)) = P(\tau \in (s,t)) = F(t) - F(s)$$

Note that when $s \mapsto F(s)$ is continuous

$$\Lambda(t) - \Lambda(s) = \int_s^t \frac{1}{1-F(r)} F(dr) = -\int_s^t d_r \log P(\tau > r) =$$
$$\log P(\tau > s) - \log P(\tau > t) = \log P(\tau > s) P(\tau > t)$$

Therefore

$$E\Bigg((\Lambda_{s\wedge\tau} - \Lambda_{s\wedge\tau})\mathbf{1}(\tau > s)\Bigg) =$$
$$\int_s^\infty \Bigg(\log(1 - F(r \wedge s)) - \log((1 - F(r \wedge t))\Bigg) F(dr) =$$
$$\log(1-F(s))(1-F(s)) - \log(1-F(t))(1-F(t)) + \int_s^t \log(1-F(r)) d_r(1-F(r))$$
$$= -\int_s^t (1-F(r)) d \log(1-F(r)) =$$
$$-\int_s^t (1-F(r)) \frac{1}{(1-F(r))} d_r(1-F(r)) = F(t) - F(s)$$

using integration by parts.