Stochastic analysis, autumn 2011, Exercises-7, 01.11.2011

1. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

   We introduce the random variables
   $X_t(\omega) = \mathbf{1}\{$ the ball drawn at time t is black $\}$
   and denote $S_t = (1 + X_1 + \cdots + X_t)$,
   $M_t = S_t/(t + 2)$, the proportion of black balls in the urn.
   We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : \ s \in \mathbb{N}, \ s \le t\}$.

   i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

   Solution

$$E(S_t|\mathcal{F}_{t-1}) = S_{t-1} + E(X_t|\mathcal{F}_{t-1}) = S_{t-1} + M_{t-1} = S_{t-1}\left(1 + \frac{1}{t+1}\right)$$

   The Doob decomposition is

$$S_t = 1 + \sum_{r=1}^{t} S_{r-1}\frac{1}{t+1} + \sum_{r=1}^{t}(X_r - M_{r-1})$$

   The predictable part is non-decreasing, we see that S_t is a submartingale.

   ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.
   Solution

$$E\big(M_t|\mathcal{F}_{t-1}\big) = \frac{1}{t+2}E(S_t|\mathcal{F}_{t-1}) = \frac{1}{t+2}\left(1 + \frac{1}{t+1}\right)S_{t-1} = \frac{S_{t-1}}{t+1} = M_{t-1}$$

   and

$$M_t - M_{t-1} = \frac{S_t}{t+2} - \frac{S_{t-1}}{t+1} = \frac{S_{t-1} + X_t}{t+2} - \frac{S_{t-1}}{t+1} = \frac{1}{t+2}\left(X_t + S_{t-1}\left(1 - \frac{t+2}{t+1}\right)\right)$$
$$= \frac{1}{t+2}\left(X_t - \frac{S_{t-1}}{t+1}\right) = \frac{1}{t+2}(X_t - M_{t-1})$$

   Therefore

$$M_t = \frac{1}{2} + \sum_{r=1}^{t} \frac{1}{r+2}(X_r - M_{r-1}) = \frac{1}{2} + (C \cdot N)_t$$

   with $C_t = \frac{1}{t+2}$ (deterministic).

iv) Note that the martingale $(M_t)_{t\geq 0}$ is uniformly integrable, why? since takes values in $[0,1]$.

Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

Solution By Doobs' martingale convergence theorem $M_t(\omega) \to M_\infty(\omega)$ P-almost surely, and by uniform integrability also in $L^1(P)$

Since M_t is uniformly integrable, $M_t = E(M_\infty|\mathcal{F}_t)$ and $E(M_\infty) = E(M_t) = E(M_0) = 1/2$.

v) Show that $P(0 < M_\infty < 1) > 0$.

Since $M_\infty(\omega) \in [0,1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.

Solution Note that if $P(M_\infty \in \{0,1\}) = 1$, then $M_\infty^2 = M_\infty$ and we have $E(M_\infty^2) = E(M_\infty)$.

Otherwise $P(M_\infty \in (0,1)) > 0$, which means $P(M_\infty^2 < M_\infty) > 0$ which implies the inequality $0 < E(M_\infty^2) < E(M_\infty)$.

By the discrete integration by parts

$$M_t^2 - M_{t-1}^2 = 2M_{t-1}(M_t - M_{t-1}) + (M_t - M_{t-1})^2$$

and since by the martingale propery

$$E\Big(2M_{t-1}(M_t - M_{t-1})\Big) = E\Big(2M_{t-1}E\big(M_t - M_{t-1}\big|\mathcal{F}_{t-1}\big)\Big) = 0$$

it follows

$$\begin{aligned}
E(M_t^2) &= \frac{1}{4} + E\Big(\sum_{r=1}^t E\big((M_r - M_{r-1})^2\big)\Big) = E(M_t^2) = \frac{1}{4} + \sum_{r=1}^t E\Big(E\big((M_r - M_{r-1})^2\big|\mathcal{F}_{r-1}\big)\Big) \\
&= \frac{1}{4} + \sum_{r=1}^t E\Big(C_r^2 E\big((X_r - M_{r-1})^2\big|\mathcal{F}_{r-1}\big)\Big) = \frac{1}{4} + \sum_{r=1}^t E\Big(C_r^2 M_{r-1}(1 - M_{r-1})\Big) \\
&= \frac{1}{4} + \sum_{r=1}^t \frac{1}{(r+2)^2}\Big(E(M_{r-1}) - E(M_{r-1}^2)\Big) = \frac{1}{4} + \sum_{r=1}^t \frac{1}{(r+2)^2}\Big(\frac{1}{2} - E(M_{r-1}^2)\Big) \\
&\leq \frac{1}{4} + \frac{1}{2}\sum_{r=1}^t \frac{1}{(r+2)^2}
\end{aligned}$$

since conditionally on $\mathcal{F}_{r-1}$, X_r is a Bernoulli variable with success probability M_{r-1}.

Actually the last inequality is strict, $E(M_t^2) > 0$ because $M_t \geq 0$ and $E(M_t) = 1/2 > 0$.

By Fatou's lemma

$$E(M_\infty^2) \le \liminf_{t\to\infty} E(M_t^2) \le \frac{1}{4} + \frac{1}{2}\sum_{r=1}^{\infty} \frac{1}{(r+2)^2} \le \frac{1}{4} + \frac{1}{2}\sum_{r=3}^{\infty} \frac{1}{r^2} < \frac{1}{4} + \frac{1}{4} = E(M_\infty)$$

where $\sum_{r=3}^{\infty} r^{-2} < \int_2^{\infty} x^{-2}dx = 1/2$ with strict inequality.

2. A branching process $(Z_t)_{t\in\mathbb{N}}$ with integer values, represents the size of a population evolving randomly in discrete time.

   We start with $Z_0(\omega) = 1$ individual at time $t = 0$.

   Inductively each of the $Z_{t-1}(\omega)$ individuals in the $(t-1)$ generation has a random number of offspring $X_{i,t}$. These offspring numbers are independent and identically distributed with law $\pi = (\pi(n) : n = 0, 1, \dots)$, $\pi(n) = P\big(X_{i,t} = n\big)$.

   The size of the new generation at time t is then

$$Z_t(\omega) = \sum_{i=1}^{Z_{t-1}(\omega)} X_{i,t}(\omega) = \sum_{i=1}^{\infty} \mathbf{1}(Z_{t-1} \ge i) X_{i,t}(\omega)$$

   We assume that the mean offspring number is finite

$$\mu = E_\pi(X) = \sum_{n=0}^{\infty} n\pi(n) < \infty$$

   - Show that $Z_t(\omega)$ is a martingale, (respectively supermartingale, submartingale) when $\mu = 1$ (respectively $0 \le \mu < 1$, $1 < \mu < \infty$, in the filtration generated by the process Z itself.

     Solution Note that

$$E(Z_t|\mathcal{F}_{t-1}) = E\left(\sum_{i=1}^{Z_{t-1}} X_{i,t}\middle|\mathcal{F}_{t-1}\right) = \sum_{i=1}^{\infty} E\left(\mathbf{1}(Z_{t-1} \le i)X_{i,t}\middle|\mathcal{F}_{t-1}\right) =$$

$$= \sum_{i=1}^{\infty} \mathbf{1}(Z_{t-1} \le i) E\left(X_{i,t}\middle|\mathcal{F}_{t-1}\right) = \sum_{i=1}^{\infty} \mathbf{1}(Z_{t-1} \le i) E(X_{i,t}) = \mu Z_{t-1}$$

     where we used independence of $X_{i,t}$ from $\mathcal{F}_{t-1}$, and by monotone convergence we can interchange sum and expectation.

   - For $\mu \ne 1$, write the Doob decomposition of Z_t and compute the mean $E(Z_t)$ for $t \in \mathbb{N}$.

     Solution

$$Z_t = 1 + \sum_{s=1}^{t} \sum_{i=1}^{Z_{s-1}} (X_{i,s} - 1) =$$

$$1 - (1-\mu)\sum_{s=1}^{t} Z_{s-1} + \sum_{s=1}^{t} \sum_{i=1}^{Z_{s-1}} (X_{i,s} - \mu)$$

and since the martingale part has zero mean

$$E(Z_t) = 1 + (\mu - 1)\sum_{s=1}^{t} E(Z_{s-1})$$

this linear difference equation has solution $E(Z_t) = \mu^t$.

- Assume that $\mu \leq 1$, and that the offspring distribution is non-trivial, meaning that $0 \leq \pi(X = 1) < 1$. The case $P(X = 1) = 1$ is trivial, nothing happens.

  Note that since $X(\omega) \in \mathbb{N}$, if $P(X = 1) < 1$ and $E(X) \leq 1$, it follows that $\pi(0) = P(X = 0) > 0$.

  Show that

$$\lim_{t\to\infty} Z_t(\omega) = 0 \qquad P \text{ a.s.}$$

Hint: first show that a finite limit $Z_\infty(\omega)$ exists P a.s.

. **Solution** Z_t is a non-negative martingale, by Doob's martingale convergence theorem it has P a.s. a finite limit Z_∞.

Consider

$$P(Z_\infty = 0|Z_1 = n) = P(Z_\infty = 0)^n$$

since $P(Z_\infty = 0)$ is the probability that descendance of a single individual becomes extinct, is the probability that independently for each of its children the respective descendances become extinct.

By computing first the conditional probability $P(Z_\infty = 0|\sigma(Z_1))(\omega)$ and taking expectation, show that the unknown $q = P(Z_\infty = 0)$ satisfies the equation

$$q = E_P(q^X), \quad q \in [\pi(0), 1]$$

where $P(X = n) = \pi(n)$ is the offspring distribution.

Note that since $\mu = E(X) \leq 1$ and $\pi(1) = P(X = 1) < 1$, necessarily $\pi(0) = P(X = 0) > 0$, and $P(Z_\infty = 0) \geq P(X = 0) > 0$. Therefore the $q = 0$ is not a solution.

$q = 1$ is a solution. We show that there are no other solutions. Note that by Jensen inequality for the concave function $x \mapsto q^x$ with $q \in [0, 1]$

$$E(q^X) \geq q^{E(X)} \geq q$$

Show that the inequality is strict in the non-trivial case with $P(X = 1) < 1$

If $0 < q < 1$ cannot be a solution since the derivative

$$\frac{d}{dq} E_P(q^X) = E\left(\frac{d}{dq} q^X\right) = E(X q^{X-1}) < E(X) \le 1$$

with strict inequality in the non-trivial case in the $P(X = 1) < 1$.

It is allowed to take a derivative inside the expectation, since $0 \le X q^{X-1} \le X \in L^1(P)$ $\forall q \in (0, 1)$.

This implies that

$$E_P(q^X) > q, \forall \quad q \in (0, 1)$$

- Assume that $\mu = 1$. Show that the martingale $(Z_t : t \in \mathbb{N})$ is not uniformly integrable.

  Solution: Since $0 = E(Z_\infty) < E(Z_t) = E(Z_0) = 1$, $Z_t(\omega) \to Z_\infty(\omega)$ P almost surely but not in $L^1(P)$, therefore uniform integrability does not hold.

3. We now make a change of measure and define a new measure P' such that under Q the offspring numbers are independent and identically distributed with

$$P'(X_{i,t} = n) = \pi'(n)$$

with $\pi'(n) = 0$ when $\pi(n) = 0$. Compute the likelihood ratio process $\frac{dP'_t}{dP_t}(\omega)$ on the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(X_{i,s} : i \in \mathbb{N}, 1 \le s \le t)$.

. Solution

$$\frac{dP'_t}{dP_t}(\omega) = \prod_{s=1}^{t} \prod_{i=1}^{Z_{s-1}(\omega)} \frac{\pi'(X_{i,s}(\omega)}{\pi(X_{i,s}(\omega)}$$

4. Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0, 1]$, $P(U_1 \in dx) = \mathbf{1}_{[0,1]}(x) dx$. Note that $E_P(U_t) = 1/2$.

Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

$$P(-\log(U_1) > x) = \begin{cases} \exp(-x) & \text{kun } x \ge 0 \\ 1 & \text{kun } x < 0 \end{cases}$$

$-\log(U_1) \in L^1(P)$ with $E_P(-\log(U_1)) = 1$.

- Let $Z_0 = 1$, and

$$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

  Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, Z_2, \ldots, Z_t) = \sigma(U_1, U_2, \ldots, U_t)$.

  Solution $E_P(Z_t | \mathcal{F}_{t-1}) = Z_{t-1} 2E(U_t | \mathcal{F}_{t-1}) = Z_{t-1} 2E(U_t) = Z_{t-1}$
  $E_P(Z_t) = E_P(Z_0) = Z_0 = 1$.

- Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.
  Solution. $(Z_t : t \in \mathbb{N})$ is a non-negative martingale, by Doob's martingale convergence theorem $\lim_{t\to\infty} Z_t(\omega)$ exists a.s.
- Show that
  $$Z_\infty(\omega) = 0 \qquad P\text{-a.s.}$$
  Hint Compute first the P-a.s. limit
  $$\lim_{t\to\infty} \frac{1}{t} \log(Z_t(\omega))$$
  (remember Kolmogorov's strong law of large numbers!). **Solution.**
  $$\begin{aligned}\frac{1}{t}\log(Z_t) &= \frac{1}{t}\left(t\log(2) + \sum_{s=1}^{t}\log(U_s)\right)\\ &= \log(2) + \frac{1}{t}\sum_{s=1}^{t}\log(U_s) \to \log(2) + E(\log(U_1)) = \log(2) - 1 < 0.\end{aligned}$$
  with convergence P a.s. and in $L^1(P)$.
  This means P a.s. as $t \to \infty$ $Z_t(\omega) = \mathcal{O}(\exp(t(\log(2)-1)) \to 0$, that is $Z_\infty(\omega) = 0$.
- Show that the martingale $(Z_t(\omega) : t \in \mathbb{N})$ is not uniformly integrable.
  Solution. Beacuse $E(Z_1) = 1 > E(Z_\infty) = 0$, so the martingale property does not hold as infinity.
- Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem ?
  Solution $\log(Z_t) = \sum_{s=1}^t \big(\log(2) + \log(U_s)\big)$ where $\log(U_s)$ are i.i.d. with $E(\log(2)+\log(U_s)) = \log(2)-1 < 0$. The assumption $\sup_t E(\log(Z_t)^-) < \infty$ of Doob convergence theorem is not satisfied, it behaves like a biased random walk, since
  $$\frac{1}{t}\log(Z_t(\omega)) = \log(2) - 1 < 0$$
  we see that P a.s. $\lim_{t\to\infty} Z_t(\omega) = -\infty$.
- At every time $t \in \mathbb{N}$, define the probability measure
  $$Q_t(A) := E_P(Z_t \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$
  on the probability space $(\Omega, \mathcal{F})$.
  Show that the random variables $(U_1, \dots, U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.
  Solution For any bounded measurable functions $f_s(x)$ $s = 1, \dots, t$
  $$\begin{aligned}&E_Q(f_1(U_1)\dots f_t(U_t)) = E_P(Z_t f_1(U_1)\dots f_t(U_t))\\ &= E_P\left(\prod_{s=1}^t 2f_s(U_s)U_s\right) = \prod_{s=1}^t 2E_P(f_s(U_s)U_s) = \prod_{s=1}^t 2\int_0^1 f_s(u)u\,du\end{aligned}$$
  which means that under Q the r.v. U_s are i.i.d. with
  $$Q(U_1 \in du) = \mathbf{1}_{[0,1]}(u)2u\,du$$