Stochastic analysis, autumn 2011, Exercises-5, 11.10.2011

1. Let $\tau_1(\omega)$ and $\tau_2(\omega)$ stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T . Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

   Solution

   $$\{\sigma \le t\} = \{\tau_1 \le t\} \cap \{\tau_2 \le t\} \in \mathcal{F}_t \quad \forall t$$

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and τ a $\mathbb{F}$-stopping time.

   Show that the stopped process $(M_{t\wedge\tau} : t \in \mathbb{R}^+)$

   $$M_t^\tau(\omega) = M_{t\wedge\tau}(\omega) = M_t(\omega)\mathbf{1}(t \le \tau(\omega)) + M_{\tau(\omega)}(\omega)\mathbf{1}(t > \tau(\omega))$$

   is a $\mathbb{F}$-martingale.

   We have shown this when $T = \mathbb{N}$ is discrete by using the martingale transform. In continuous time we have not yet defined such martingale transforms. Prove the statement directly by using the definitions.

   Solution Proposition 17, Chapter 5 in the lecture notes.

3. Let $(M_t(\omega))_{t\in T}$ a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Define the family of random times $\tau_x : x \in \mathbb{R}$

   $$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \ge x\} & \text{for } x \ge 0 \\ \inf\{s : M_s \le x\} & \text{for } x < 0 \end{cases}$$

   Show that τ_x is a stopping time.

   Solution For $x > 0, t \in T$,

   $$\{\tau_x > t\} = \{\sup_{s\le t} M_s < x\} \in \mathcal{F}_t$$

   because by the regularization lemma 21, chapter 5 in the lecture notes a martingale indexed by $\mathbb{R}^+$ has almost surely left and right limits at all times, which means

   $$\sup_{s\le t} M_s(\omega) = \sup_{s\in[0,t]\cap\mathbb{Q}} M_s(\omega)$$

   is $\mathcal{F}_t$-measurable.

   For $x < 0$ take the infimum.

4. Let

   $$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2$$

X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

- Show that $(M_t)_{t \in \mathbb{N}}$ and $(M_t^2 - t)_{t \in \mathbb{N}}$ are $\mathbb{F}$-martingales.
  Solution

$$E(M_t | \mathcal{F}_{t-1}) = E_P(M_{t-1} + X_t | \mathcal{F}_{t-1}) = M_{t-1} + 0,$$
$$E(M_t^2 | \mathcal{F}_{t-1}) = E_P((M_{t-1} + X_t)^2 | \mathcal{F}_{t-1}) =$$
$$E_P(M_{t-1}^2 + X_t^2 + 2M_{t-1}X_t | \mathcal{F}_{t-1}) = M_{t-1}^2 + E(X_t^2) + 2M_{t-1}E(X_t) = M_{t-1}^2 + 1$$

- Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t \in \mathbb{N}}$ and $(M_{t\wedge\sigma}^2 - t \wedge \sigma)_{t \in \mathbb{N}}$.
  Show that Doob's martingale convergence theorem applies and

$$\lim_{t \to \infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

  exists P-almost surely.
  Solution The stopped process $(M_{t\wedge\sigma})$ is a martingale taking values in $[a, b]$, therefore it is uniformly integrable (because it is bounded). In particlar it is bounded in $L^1(P)$ and Doob's martingale convergence theorem applies, P-almost surely (and by UI also in $L^1(P)$)

$$\lim_{t \to \infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

  By the way, this implies $P(\sigma < \infty) = 1$, since on the set $\{\sigma = \infty\}$ the random walk would continue fluctuating with $(\limsup M_t - \liminf M_t) \geq 1$. Anyway we will see also in the next step that σ is finite.

- Consider now $(M_{t\wedge\sigma}^2 - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.
  The stopped martingale $M_{t\wedge\sigma}^2 - t \wedge \sigma$ is a submartingale bounded from above, since

$$M_{t\wedge\sigma}^2 - t \wedge \sigma \leq (a^2 \vee b^2)$$

  By Doob's convergence theorem it has a limit P a.s. and since $M_{t\wedge\sigma}(\omega) \to M_\sigma(\omega)$ with $\sigma(\omega) < \infty$

$$\lim_{t \to \infty} M_{t\wedge\sigma}^2 - t \wedge \sigma = M_\sigma^2 - \sigma \in L^1(P)$$

  P almost surely,
  By the reverse Fatou lemma,
  which applies since $M_{t\wedge\sigma}^2 - t \wedge \sigma \leq (a^2 \vee b^2)$,

$$E(M_\sigma^2 - \sigma) = E(\limsup_{t \to \infty} M_{t\wedge\sigma}^2 - t \wedge \sigma) \geq \limsup_{t \to \infty} E(M_{t\wedge\sigma}^2 - t \wedge \sigma) = 0$$

  which implies

$$E(\sigma) \leq E(M_\sigma^2) \leq a^2 \vee b^2 < \infty$$

- For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.

  Solution

  $$0 = E(M_{t\wedge\sigma}) = E(M_\sigma)$$

  where $P(\sigma < \infty) = 1$ and by uniform integrability we can take the limit as $t \to \infty$ inside the expactation. Now

  $$0 = E(M_\sigma) = aP(M_\sigma = a) + bP(M_\sigma = b) = aP(\tau_a < \tau_b) + b\big(1 - P(\tau_a < \tau_b)\big)$$
  $$\implies P(\tau_a < \tau_b) = \frac{b}{b-a}$$

5. Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$

   - Show that B_t and $(B_t^2 - t)$ are $\mathbb{F}$-martingales.
     Solution It follows since $(B_t - B_s) \perp\!\!\!\perp \mathcal{F}_s$ for $0 \le s \le t$.

     $$E(B_t^2|\mathcal{F}_s) = B_s^2 + E((B_t - B_s)^2|\mathcal{F}_s) + 2B_sE(B_t - B_s|\mathcal{F}_s)$$
     $$B_s^2 + E((B_t - B_s)^2) + 2B_sE(B_t - B_s) = B_s^2 + (t - s) + 0$$

     Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P(\sigma < \infty) = 1$.
   - Let $a < 0 < b \in \mathbb{R}$. Compute $P(\tau_a < \tau_b)$,

   It follows by Proposition 17, Chapter 5 in the lecture notes that the stopped process $B_{t\wedge\sigma}(\omega)$ is a martingale, and since it takes values in $[a, b]$ it is bounded and therefore uniformly integrable.

   To compute $P(\tau_a < \tau_b)$, use first the martingale property and the bounded convergence theorem

   $$0 = E(B_0)E(B_{t\wedge\sigma}) \to E(B_\sigma) = aP(B_\sigma = a) + b(1 - P(B_\sigma = a))$$

   and we obtain the same result as in the random walk case.