Stochastic analysis, autumn 2011, Exercises-4, 4.10.11

1. Show:

   - For $X, Y \in L^2(P)$,
   $$\| X+Y \|_2^2 + \| X-Y \|_2^2 = 2 \| X \|_2^2 + 2 \| Y \|_2^2$$
   (parallelogram identity, suunnikkaan identiteetti)

   - $$(X,Y)_{L^2(P)} = E_P(XY) = \frac{1}{4}\big(\| X+Y \|_2^2 - \| X-Y \|_2^2\big)$$
   (Polarization identitity, polarisaation identiteetti)

   - A norm $\| x \|$ satisifies the parallelogram identity if and only if the polarization identity defined a scalar product (x,y) (bilinear, symmetric and positive) such that $\| x \|^2 = (x,x)$.

   Solution (from Yosida, Functional analysis, pp.39) Assume that the norm $\| \cdot \|$ satisfies the parallelogram identity and define the scalar product by polarization:
   $$(X,Y) := \frac{1}{4}\big(\| X+Y \|_2^2 - \| X-Y \|_2^2\big)$$
   Then
   $$\begin{aligned}
   &(X,Z)+(Y,Z)\\
   &= \frac{1}{4}\big(\| Z+X \|_2^2 - \| Z-X \|_2^2 + \| Z+Y \|_2^2 - \| Z-Y \|_2^2\big)\\
   &= \frac{1}{4}\Bigg(\big\{\| Z+(X+Y)/2+(X-Y)/2 \|_2^2 + \| Z+(X+Y)/2-(X-Y)/2 \|_2^2\big\}\\
   &-\big\{\| Z-(X+Y)/2-(X-Y)/2 \|_2^2 + \| Z-(X+Y)/2+(X-Y)/2 \|_2^2\big\}\Bigg)
   \end{aligned}$$
   and by the parallelogram identity
   $$\begin{aligned}
   &= \frac{1}{4}\Bigg(2\big\{\| Z+(X+Y)/2 \|_2^2 + \| (X-Y)/2 \|_2^2\big\}\\
   &-2\big\{\| Z-(X+Y)/2 \|_2^2 + \| (X-Y)/2 \|_2^2\big\}\Bigg)\\
   &= \frac{1}{2}\Bigg(\| Z+(X+Y)/2 \|_2^2 - \| Z-(X+Y)/2 \|_2^2\Bigg)\\
   &= 2\big(\frac{(X+Y)}{2}, Z\big)
   \end{aligned}$$
   When $Y=0$, $(0,Z) = (\| Z+0 \|^2 - \| Z-0 \|^2)/4 = 0$, which implies
   $$(X,Z) = 2\big(\frac{X}{2}, Z\big) = 2^n\big(2^{-n}X, Z\big) \quad \text{kun } n \in \mathbb{Z}$$

and

$$(X,Z) + (Y,Z) = (X+Y,Z)$$

It follows that

$$(X,Z) = k2^{-n}(k2^{n}X,Z), \quad \forall k \in \mathbb{N}, n \in \mathbb{Z}$$

which means

$$(X,Z) = q(q^{-1}X,Z)\forall q \in Q$$

Since in the normed space the maps $q \to \| qX + Z \|$ ja $q \to \| qX - Z \|$ are continuous , it follows

$$(X,Z)r = (rX,Z), \quad \forall r \in \mathbb{R}$$

2. Consider random variable $X(\omega)$ on $(\Omega, \mathcal{F}, P)$, assume either

   - **a)** $P(X \in dx) = \mathbf{1}(x \geq 0)\exp(-x)dx$, X has 1-exponential distribution.

     or
   - **b)** $P(X \in dx) = \pi^{-1}(1+x^2)^{-1}dx$, X has Cauchy distribution.

   Show that under **a)** $E_P(|X|) < \infty$ while under **b)** $E_P(|X|) = \infty$

   Solution. a) by parts

$$E_P(|X|) = E_P(X) = \int_0^\infty x\exp(-x)dx = \Big[-x\exp(-x) \Big]_0^\infty + \int_0^\infty \exp(-x)dx = 0 + 1 < \infty$$

   Solution. b)

$$E_P(|X|) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx \geq \frac{2}{\pi}\int_1^\infty \frac{x}{1+x^2}dx \geq \frac{2}{\pi}\int_1^\infty \frac{1}{x}dx = +\infty$$

   Let $Y(\omega) = \lfloor X(\omega) \rfloor = \max\{n \in \mathbb{Z} : n \leq X(\omega)\} \in \mathbb{Z}$.

   Show under **a**) and under **b**) the conditional expectation

$$E_P(X|\sigma(Y))(\omega) \in \mathbb{R}$$

   Hint Note that theσ-algebra $\sigma(Y)$ is countably generated, by a countable $\mathcal{F}$-measurable partition of Ω. In this case the values taken by the conditional expectation (which is a random variable) are elementary conditional expectations obtained by conditioning on events of positive probability.

   Solution.

$$E_P(X|\sigma(Y))(\omega) = \sum_{z\in\mathbb{Z}} \frac{E_P(X\mathbf{1}(z \leq X < z+1))}{P(z \leq X \leq z+1)}\mathbf{1}(z \leq X(\omega) < z+1)$$

   where the series contains only one nonzero term.

   We see also that $P(z \leq X \leq z+1) > 0 \ \forall z \in \mathbb{Z}$ and

$$z \leq \frac{E_P(X\mathbf{1}(z \leq X < z+1))}{P(z \leq X \leq z+1)} < (z+1)$$

3. Let $X(\omega)$ and $Y(\omega)$ P-independent and identically distributed on $[0,1]$:

$$P(X \in dx, Y \in dy) = \mathbf{1}_{[0,1]}(x)\ \mathbf{1}_{[0,1]}(y)\ dx\ dy$$

Let $Z(\omega) = \min(X(\omega), Y(\omega))$.

Compute the conditional expectation $E_P(X|\sigma(Z))(\omega)$.

Vihje It may be easier take first conditional expectation with respect to a larger σ-algebra:

$$E_P(Z|\sigma(Z)) = E_P\big(E_P(X|\sigma(Z,I))\big|\sigma(Z)\big)$$

with $I(\omega) := \mathbf{1}(X(\omega) \le Y(\omega))$ and $\sigma(Z,I) \supseteq \sigma(Z)$.

Solution. Let's compute first

$$E_P(X|\sigma(Z,I))(\omega)$$

Since

$$X(\omega) = X(\omega)I(\omega) + X(\omega)(1-I(\omega)) = Z(\omega)I(\omega) + X(\omega)(1-I(\omega))$$

we have

$$\begin{aligned}&E_P(X|\sigma(Z,I)) = E_P(ZI|\sigma(Z,I)) + E_P(X(1-I)|\sigma(Z,I))\\&= ZI + E_P(X(1-I)|\sigma(Z,I))\end{aligned}$$

Since

$$Z(\omega)(1-I(\omega)) = Z(\omega)\mathbf{1}(Y(\omega) < X(\omega)) = Y(\omega)(1-I(\omega))$$

it follows $\forall B \in \mathcal{B}(\mathbb{R})$,

$$E_P\big(X(1-I)\mathbf{1}(Z \in B)) = E_P\big(X(1-I)\mathbf{1}(Y \in B))$$

and by the definition of conditional expectation

$$\begin{aligned}&E_P\big(E_P(X|\sigma(Z,I))(1-I)\mathbf{1}(Z \in B)\big) = E_P\big(E_P(X|\sigma(Z,I))(1-I)\mathbf{1}(Y \in B)\big)\\&= E_P\big(E_P(X|\sigma(Y,I))(1-I)\mathbf{1}(Y \in B)\big)\end{aligned}$$

This gives

$$E_P(X|\sigma(Z,I))(\omega)(1-I(\omega)) = E_P(X|\sigma(Y,I))(\omega)(1-I(\omega))$$

Since $I(\omega) \in \{0,1\}$

$$E_P(X|\sigma(Y,I))(\omega)(1-I(\omega)) = (1-I(\omega))\frac{E_P(X\mathbf{1}(X>Y)|\sigma(Y))(\omega)}{E_P(\mathbf{1}(X>Y)|\sigma(Y))(\omega)}$$

and X, Y are P-independent,

$$\begin{aligned}&= \frac{E_P(X\mathbf{1}(X>y))}{P(X>y)}\bigg|_{y=Y(\omega)}(1-I(\omega)) = \frac{\int_{Y(\omega)}^1 u du}{1-Y(\omega)}(1-I(\omega)) =\\&= \frac{1}{2}\frac{(1-Y(\omega)^2)}{(1-Y(\omega))}(1-I(\omega))\\&= \frac{1}{2}(1+Y(\omega))\mathbf{1}(Z(\omega)=Y(\omega)) = \frac{1}{2}(1+Z(\omega))(1-I(\omega))\end{aligned}$$

Therefore

$$E_P(X|\sigma(Z,I))(\omega) = Z(\omega)I(\omega) + \frac{1}{2}(1+Z(\omega))(1-I(\omega))$$

Taking conditional expectation,

$$E_P(X|\sigma(Z))(\omega) = E_P(ZI|\sigma(Z))(\omega) + \frac{1}{2}E_P\big((1+Z)(1-I)|\sigma(Z)\big)(\omega) =$$
$$Z(\omega)P(X \le Y|\sigma(Z))(\omega) + \frac{1}{2}(1+Z(\omega))P(X > Y|\sigma(Z))(\omega) =$$
$$= \frac{1}{2}Z(\omega) + \frac{1}{4}(1+Z(\omega)) = \frac{1}{4} + \frac{3}{4}Z(\omega)$$

and by symmetry we see that $P(X > Y|\sigma(Z))(\omega) \equiv \frac{1}{2}$.

4. Using the definition of conditional expectation together with Fatou's lemma for expectations,

   prove Fatou's lemma for conditional expectations: if $0 \le X_n(\omega)$, $\forall n \in \mathbb{N}$

$$0 \le E_P(\liminf X_n|\mathcal{G})(\omega) \le \liminf_n E_P(X_n|\mathcal{G})(\omega)$$

   Solution.

$$\liminf X_n(\omega) = \sup_{n\ge 0} \inf_{k \ge n} X_k(\omega)$$

   where $\inf_{k\ge n} X_k(\omega) \uparrow \liminf X_n(\omega)$ as $n \uparrow \infty$ Now by monotone convergence of conditional expectation

$$\inf_{k\ge n} E(X_k|\mathcal{G}) \le E(\inf_{k\ge n} X_k|\mathcal{G}) \uparrow E(\liminf_n X_n|\mathcal{G})$$

   and Fatou lemma follows since the left hand side converges monotonically to $\liminf_n E(X_n|\mathcal{G})$ as $n \uparrow \infty$

5. Let $X, Y \in L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra Check the identity

$$\mathrm{Cov}_P(X,Y) = E_P\big(\ \mathrm{Cov}(X,Y|\mathcal{G})\big) + \mathrm{Cov}_P\big(E_P(X|\mathcal{G}), E_P(Y|\mathcal{G})\big)$$

   when $\mathcal{G} = \sigma(X)$ or $\mathcal{G} = \sigma(Y)$.

Solution Since adding constant term does not change the covariance, we assume $E(X) = E(Y) = 0$. By the polarization identity it is enough to check the formula when $X = Y$.

$$E(X^2) \overset{?}{=} E\Big(\big(X - E(X|\mathcal{G})^2|\mathcal{G}\big)\Big) + E\Big(E(X|\mathcal{G})^2\Big)$$

which follows by adding and subtracting:

$$E\Big(\Big(E(X|\mathcal{G}) + \big(X - E(X|\mathcal{G})\big)\Big)^2\Big) =$$
$$E(E(X|\mathcal{G})^2 + E\Big(\big(X - E(X|\mathcal{G})\big)^2\Big) = +2E\Big(E(X|\mathcal{G})\big(X - E(X|\mathcal{G})\big)\Big)$$

where the cross term has 0 expectation by the property of the conditional expectation.