Stochastic analysis, autumn 2011, Exercises-2, 23.09.11

1. Let

$$f(x) = f(0) + \int_0^x \dot{f}(y)dy, \quad h(x) = h(0) + \int_0^x \dot{h}(y)dy,$$

absolutely continous function with $\dot{f}, \dot{h} \in L^2(\mathbb{R}, \gamma)$ where $\gamma(dx) = \phi(x)dx$ denotes the standard gaussian measure.

- For a standard gaussian random variable $G(\omega)$ with $E(G) = 0,\ E(G) = 1$ prove the gaussian integration by parts formula:

$$E_P\Big(f'(G)h(G) \Big) = E_P\Big(f(G)(Gh(G) - h'(G)) \Big)$$

Hint: rewrite the expectation as integral, and use integration by parts with respect to Lebesgue measure.
We show first that $E(f'(G)^2) < \infty$ implies $E(f(G)^2) < \infty$.

$$E\Big((f(G) - f(0))^2 \mathbf{1}(G \geq 0) \Big) \int_0^\infty \Big(\int_0^x f'(y)dy \Big)^2 \phi(x)dx$$
$$\leq \int_0^\infty \int_0^x f'(y)^2 \phi(x)dydx$$
$$= \int_0^\infty \Big(\int_y^\infty \phi(x)dx \Big) f'(y)^2 dy \leq \int_0^\infty f'(y)^2 \min(1, \frac{1}{y})\phi(y)dy$$
$$\leq E(f'(G)^2 \mathbf{1}(G \geq 0))$$

where we have used Cauchy-Schwartz, Fubini, and a gaussian tail bound. Similarly

$$E((f(G) - f(0))^2 \mathbf{1}(G \leq 0)\Big) \leq E(f'(G)^2 \mathbf{1}(G \geq 0))$$

which implies by the triangle ineguality

$$E(f(G)^2)^{1/2} \leq E(f'(G)^2)^{1/2} + |f(0)| < \infty$$

Therefore it is enough to assume $f'(G), h'(G) \in L^2(\mathbb{R}, \gamma)$.
Integrating by parts,

$$f(b)h(b)\phi(b) - f(a)h(a)\phi(a) =$$
$$\int_a^b f'(x)h(x)\phi(x)ds + \int_a^b f(x)h(x)'\phi(x)ds + \int_a^b f(x)h(x)\phi(x)'ds$$
$$= \int_a^b f'(x)h(x)\phi(x)ds + \int_a^b f(x)h(x)'\phi(x)ds - \int_a^b xf(x)h(x)\phi(x)ds$$

since for the standard gaussian density $\phi'(x) = -x\phi(x)$. We show that $f(x)h(x)\phi(x) \to 0$ as $x \to \pm\infty$. Since

$$\int_0^\infty |f(x)h(x)|\phi(x)dx \leq \sqrt{E(f(G)^2)E(h(G^2))} < \infty$$

we have

$$\lim_{b\to\infty}\int_b^\infty |f(x)h(x)|\phi(x)dx = 0$$

and since $f(b)h(b)$ is an absolutely continuous function with integrable derivative

$$E(|f'(G)h(G)) + f(G)h'(G)|) \le E(|f'(G)h(G)|) + E(|h'(G)h(G)|) < \infty$$

necessarily

$$\lim_{b\to\infty} f(b)h(b)\phi(b) = 0$$

otherwise there is $x_n \to \infty$ such that $|f(x_n)h(x_n)\phi(x_n)| > \varepsilon$. But $\forall \eta > 0$ and $\forall n$ large enough, $r > 0$

$$\phi(x_n + r)\int_{x_n}^{x_n+r} |\frac{d}{dy}(fh)(y)|dy \le \int_{x_n}^{x_n+r} |\frac{d}{dy}(fh)(y)|\phi(y)dy \le \int_{x_n}^{\infty} |\frac{d}{dy}(fh)(y)|\phi(y)dy < \eta$$

which gives

$$|f(x_n + r)h(x_n + r)| > \frac{\varepsilon}{\phi(x_n)} - \frac{\eta}{\phi(x_n + r)}.$$

This gives

$$\int_{x_n}^{x_n+\Delta} |f(x)h(x)|\phi(x)dx > \int_0^\Delta \left(\varepsilon\frac{\phi(y)}{\phi(x_n)} - \eta\right)dy$$
$$= \varepsilon\int_0^\Delta \exp(-\frac{r^2}{2} + 2rx_n)dr - \eta\Delta \to \infty \text{ as } x_n \to \infty$$

for fixed $\eta > 0$ and $\Delta > 0$ This is in contradiction with $E(|f(G)h(G)|) < \infty$.

By taking limits as $a \to -\infty$, $b \to +\infty$ we obtain

$$0 = \int_{\mathbb{R}} f'(x)h(x)\phi(x)ds + \int_{\mathbb{R}} f(x)h(x)'\phi(x)ds - \int_{\mathbb{R}} xf(x)h(x)\phi(x)ds =$$
$$E(f'(G)h(G)) + E(f(G)(h'(G) - Gh(G)))$$

- Write the corresponding gaussian integration by parts for $B_T(\omega) \sim \mathcal{N}(0, T)$ with $T > 0$
Solution Take $B_T(\omega) = \sqrt{T}G(\omega)$ with $G(\omega) \sim \mathcal{N}(0, 1)$
Then

$$E_P(f'(B_T)h(B_T)) = E_P(f'(\sqrt{T}G)h(\sqrt{T}G)) = E_P\left(\frac{1}{\sqrt{T}}\frac{d}{dx}f(\sqrt{T}x)\Big|_{x=G} h(\sqrt{T}G)\right)$$
$$= \frac{1}{\sqrt{T}}E_P\left(f(\sqrt{T}G)\left(h(\sqrt{T}G)G - \frac{d}{dx}h(\sqrt{T}x)\Big|_{x=G}\right)\right)$$
$$= E_P\left(f(\sqrt{T}G)(\frac{h(\sqrt{T}G)\sqrt{T}G}{T} - h'(\sqrt{T}G))\right)$$
$$= E_P\left(f(B_T)(\frac{h(B_T)B_T}{T} - h'(B_T))\right)$$

2. Show that a process X_t with independent increments is Markov, which means for $0 \leq s \leq t$ and$\mathcal{F}_s^X = \sigma(X_u : 0 \leq u \leq s)$ and a bounded measurable test function $f(x)$

$$E(f(X_t)|\mathcal{F}_s^X) = E(f(X_t)|\sigma(X_s))$$

Hint. use first the decomposition $X_t = X_s + (X_t - X_s)$ and the definition of conditional expectation to show first that for a bounded measurable test-function

$$E(f(X_t)|\sigma(X_s))(\omega) = E\bigg(f(y + (X_t - X_s))\bigg)\bigg|_{y=X_s(\omega)}$$

Solutions We will use the following lemma: if X and Y are random variables on $(\Omega, \mathcal{F}, P)$, $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-algebra and Y is $\mathcal{G}$-measurable, and X is P-indepenent from the σ-algebra $\mathcal{G}$,then for every non-negative Borel-measurable function $f(x, y)$, and X

$$E_P(f(X,Y)|\mathcal{G})(\omega) = E_P(f(X,y))\bigg|_{y=Y(\omega)}$$

Proof: it is true for $f(x, y) = f_1(x)f_2(y)$, by using independence and the definition of conditional expectation. every non-negative Borel-measurable function $f(x, y)$ can be approximated from below by finite linear combinations of such product functions and the results follows by monotone convergence of conditional expectation.

Now for $0 \leq s \leq t$

$$E(f(X_t)|\mathcal{F}_s^X) = E_P(f(X_s + (X_t - X_s))|\mathcal{F}_s^X)$$
$$= E_P(f(y + X_t - X_s))\bigg|_{y=X_s} = E_P(f(X_s + (X_t - X_s))|\sigma(X_s)) = E_P(f(X_t)|\sigma(X_s))$$

where we used the previous lemma twice.

3. $(N_t(\omega) : t \geq 0)$ is a λ-Poisson process ($\lambda > 0$) if $N_0 = 0$ and it has independent increments with $(N_t - N_s) \sim \text{Poisson}(\lambda(t - s))$

   - Show that the Poisson process is non-decreasing with piecewise constant trajectories, it is P-almost surely finite on and increases only by jumps of size 1.

     Hint: Show that

     $$P\bigg(\exists t \leq T : \Delta N_t \geq 2\bigg) = 0$$

     Write it as limits of probabilities of events depending on a finite number of increments.

     Solution: It is clear that since the increments take value in ÂăN, that the Poisson process is piecewise constant, takes values in $\mathbb{N}$, it is non-decreasing and piecewise constant increasing only by jumps.

We show that jumps have size 1.

Consider the dyadics, $D_n = \{k2^{-n}, k \in \mathbb{N}\}, D = \bigcup_{n\in\mathbb{N}} D_n$.
Let

$$A_k^n = \{\omega : (N_{k2^{-n}} - N_{(k-1)2^{-n}}) \leq 1\}$$

$$P(A_k^n) = \exp(-\lambda 2^{-n})\Big(1 + \lambda 2^{-n}\Big)$$

Let also

$$A^n = \bigcap_{k=1}^{2^n} A_k^n$$

By the independence of the increments

$$P(A^n) = \prod_{k=1}^{2^n} P(A_k^n) = \exp(-\lambda 2^{-n})^{2^n}\Big(1 + \lambda 2^{-n}\Big)^{2^n}$$
$$= \exp(-\lambda)\Big(1 + \lambda 2^{-n}\Big)^{2^n} \uparrow \quad \exp(-\lambda)\exp(\lambda) = 1$$

as $n \to \infty$.
Moreover there is a subsequence n_m such that

$$0 \leq (1 - P(A_{n_m}) \leq 2^{-m}$$

which implies

$$\sum_{m=0}^{\infty} P\big((A_{n_m})^c\big) < \infty$$

by Borel Cantelli lemma

$$P(\liminf_m A^{n_m}) = 1 - P(\limsup_m (A^{n_m})^c) = 1$$

which means that P a.s., for all m large enough, for all $k = 1, \dots, 2^{n_m}$

$$(N_{k2^{-n_m}} - N_{(k-1)2^{-n_m}}) \leq 1$$

But since increments are non-negative this means P-almost surely and $\forall d \in D_n \cap [0,1]$ with n large enough

$$(N_{d+2^{-n}} - N_d) \leq 1$$

But this implies that P-almost surely, for all $t \in [0,1]$

$$0 \leq N_{d+}(\omega) - N_{d-}(\omega) \leq 1$$

where

$$N_{t+}(\omega) = \lim_{d\downarrow t, t\in D} N_d(\omega), \quad N_{t-}(\omega) = \lim_{d\uparrow t, t\in D} N_d(\omega)$$

- Compute the probability density of the first jump time $\tau(\omega)$. $P(\tau > t) = P(N_t = 0) = \exp(-\lambda t)$

$$p_\tau(t) = -\frac{d}{dt}P(\tau > t) = \exp(-\lambda t)\lambda$$

4. For a fixed $t \in [0,1]$ consider the function $s \mapsto h_t(s) := (s \wedge t) = \min(s,t)$

    - Show that $h_t(\cdot) = (t \wedge \cdot)$ belongs to the Cameron-Martin space H.
    Solution

    $$t \wedge s = \int_0^s \mathbf{1}(u \le t)du$$

    where $u \mapsto \mathbf{1}(u \le t)$ is square integrable because it is bounded.
    - Show that $B_t(\omega) := B(h_t)$ is a Brownian motion.
    - Show that $K(t,s) := \mathrm{Cov}(B_t, B_s) = E(B_t B_s) = s \wedge t$.
    The covariance of

    $$E(B_t B_s) = (h_t, h_s)_H = \int_0^1 \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)ds = t \wedge s$$

    and

    $$E((B_t - B_s)(B_v - B_u)) = t \wedge v + u \wedge s - t \wedge u - s \wedge v$$

    in case $0 \le s \le t \le u \le v$ we get $(t + s - t - s) = 0$, so disjoint increments are uncorrelated. In case $t = v$ $s = v$ we get $(t - s)$.
    By definition of isonormal gaussian process the variables $(B_{t_1}, \dots, B_{t_d}) = (B(h_{t_1}), \dots, B(h_{t_d}))$ are jointly gaussian with zero mean and so are the increments, $(\Delta B_i = (B_{t_i} - B_{t_{i-1}}), i = 1, \dots, d)$, since they are obtained by a linear tranformation.
    Therefore uncorrelated jointly gaussian increments are independent.
    By isometry we obtain the series expansion with respect to the Haar basis

    $$B_t = \sum_{n \in \mathbb{N}} \sum_{d \in D_n \setminus D_{n-1}} B(\eta_d)(\eta_d, (\cdot \wedge t))_H =$$
    $$\sum_{n \in \mathbb{N}} \sum_{d \in D_n \setminus D_{n-1}} B(\eta_d)(\dot{\eta}_d, \mathbf{1}_{[0,t]})_H =$$
    $$\sum_{n \in \mathbb{N}} \sum_{d \in D_n \setminus D_{n-1}} B(\eta_d) \int_0^t \dot{\eta}_d(s)ds$$
    $$= \sum_{n \in \mathbb{N}} \sum_{d \in D_n \setminus D_{n-1}} B(\eta_d)\eta_d(t)$$

    where the series converges in $L^2(\Omega)$. In the Lévy construction we have shown also that the expansion converges P almost surely uniformly in $[0,1]$, which implies P-a.s.continuity.

- Show the reproducing kernel Hilbert space property in the Cameron Martin space H: for $h \in H$

$$\big(K(t,\cdot), h(\cdot)\big)_H = h(t)$$

Sol. We have seen that $K(t,s) = E(B_t B_s) = (t \wedge s)$

$$\big(K(t,\cdot), h(\cdot)\big)_H = ((t \wedge \cdot), h)_H =$$

$$(\mathbf{1}_{[0,t]}, \dot{h})_{L^2([0,1])} = \int_0^t \dot{h}(s) ds = h(t)$$