Stochastic analysis, autumn 2011, Exercises-10, 22.11.2011

1. Let τ be a $\mathbb{F}$-stopping time in the filtration generated by a Brownian motion B_t, such that $E(\tau) < \infty$.

   - Use Doob maximal inequality to show that $(B_{\tau\wedge t} : t \in \mathbb{R}^+)$ is a martingale bounded in $L^2(P)$.

     Solution

     $$\sup_{t\geq 0} E\big(B^2_{\tau\wedge t}\big) = \sup_{t\geq 0} E\big(\langle B\rangle_{t\wedge\tau}\big) = \sup t > 0 E(t\wedge\tau) = E(\tau) < \infty$$

     by monotone convergence.

   - Prove Wald's identities

     $$E(B_\tau) = 0 \quad , \quad E(B^2_\tau) = E(\tau)$$

     Hint: Doob optional sampling theorem cannot be applied directly since $(B_t : t \in \mathbb{R}^+)$ is not uniformly integrable, neither $\tau(\omega)$ is assumed to be bounded. Note also that

     $$B_\tau(\omega) = \sum_{n=1}^{\infty}\big(B_{\tau\wedge n}(\omega) - B_{\tau\wedge(n-1)}(\omega)\big)$$

     Solution Note that

     $$E\Bigg(B_{\tau\wedge n}(\omega)^2\Big) = E\Bigg(\Bigg\{\sum_{k=1}^{n}\big(B_{\tau\wedge n}(\omega) - B_{\tau\wedge(n-1)}(\omega)\big)\Bigg\}^2\Bigg) =$$
     $$\sum_{k=1}^{n} E\Bigg((B_{\tau\wedge k}(\omega) - B_{\tau\wedge(k-1)})^2\Bigg)$$
     $$+2\sum_{k=1}^{n}\sum_{0\leq h<k} E\Bigg((B_{\tau\wedge k}(\omega) - B_{\tau\wedge(k-1)})E\big(B_{\tau\wedge k}(\omega) - B_{\tau\wedge(k-1)}\big|\mathcal{F}_{k-1}\big)\Bigg)$$
     $$\sum_{k=1}^{n} E(\langle B\rangle_{\tau\wedge k} - \langle B\rangle_{\tau\wedge(k-1)}\Bigg) = \sum_{k=1}^{n} E\big(\tau\wedge k - \tau\wedge(k-1)\big) =$$
     $$E(\tau\wedge n) \uparrow E(\tau) \text{ as } n\uparrow\infty$$

     by the martingale property of $(B_{\tau\wedge n})$ in the filtration $(\mathcal{F}_{\tau\wedge n} : n \in \mathbb{N})$ which follows by Doob's optional sampling theorem for bounded stopping times. Moreover $\lim_{n\to\infty} B_{\tau\wedge n} = B_\tau$ almost surely and in $L^2(P)$ sense, which implies $E(B^2_\tau) = E(\tau)$

     This means $\sup_n E(B^2_{\tau\wedge n}) \leq E(\tau) < \infty$, the collection of r.v. $\{B_{\tau\wedge n} : n \in \mathbb{N}\}$ is bounded in L^2 and therefore it is uniformly integrable. Therefore

     $$0 = E(B_0) = E(B_{\tau\wedge n}) \to E(B_\tau) \text{ as } n\to\infty$$

2. Let M_t a continuous $\mathbb{F}$-martingale with $E(M_t^2) < \infty \ \forall t$, and let A_t be a continuous and bounded $\mathbb{F}$-adapted process with finite variation on finite intervals.

   Show that for $0 \leq s \leq t$

$$M_t A_t - M_s A_s = \int_0^t A_s dM_s + \int_0^t M_s dA_s$$

   where on the right side we have an Ito integral and a Riemann Stieltjes integral.

   Note that the Ito integral $((A \cdot M)_t : t \in \mathbb{R}^+)$ is a square integrable martingale (why ?).

   Hint Note that

$$M_t A_t - M_s A_s = M_t(A_t - A_s) + A_s(M_t - M_s)$$

   and use telescopic sums for some $s = r_0 < r_1 < \dots < r_n = t$, letting the step-size of the partition going to zero.

   Solution By using telescopic sums, for $0 < t_1 < t_2 < \dots$

$$M_t A_t - M_0 A_0 = \sum_{t_i} M_{t_i}(A_{t \wedge t_i} - A_{t \wedge t_{i-1}}) + \sum_{t_i} A_{t_i}(M_{t \wedge t_i} - M_{t \wedge t_{i-1}})$$

   as $\Delta = \sup_i (t_i - t_{i-1}) \to 0$, P a.s. the first sum converges to the Riemann Stieltjes integral $\int\limits_0^t M_s dA_s$ since P a.s. $A_t(\omega)$ has finite variation on compacts and $M_t(\omega)$ is continuous.Since A_t is bounded and M_t is square integrable by the bounded convergence theorem we have also convergence in $L^2(P)$.

   The second sum converges is the limit of Ito integrals $\int\limits_0^t A_s^\Pi dM_s$ with simple predictable integrands, since

$$A_s^\Pi := A_{t_i}(\omega) \mathbf{1}_{(t_{i-1}, t_i]}(s) \to A_s \text{Âă in } L^2(\Omega \times [0,t], \mathcal{P}, dP \otimes d\langle M \rangle),$$

   $\int\limits_0^t A_s^\Pi dM_s \longrightarrow \int A_s dM_s$ in $L^2(\Omega)$.

3. Let B_t a Brownian motion and denote by $\mathbb{F}$ its filtration.

   Consider the pathwise Ito-Föllmer formula.

$$f(B_t, t) = f(B_0, 0) + \int_0^t f_x(B_s, s) dB_s + \int_0^t \left(f_s(B_s, s) + \frac{1}{2} f_{xx}(B_s, s) \right) ds$$

   where the pathwise Föllmer integral coincides with the Ito integral. We assume that $f(x, s)$ is such that the the integrals above exist. Since the gaussian distribution has exponential moments, it is more than enough to assume that the derivatives have polynomial growth.

   Show that if $f(B_t, t)$ is a local martingale, necessarily

$$f(B_t, t) = f(B_0, 0) + \int_0^t f_x(B_s, s) dB_s$$

Hint: a local martingale with finite variation is constant.

Solution By Ito formula the difference

$$f(B_t,t)-f(B_0,0)-\int_0^t f_x(B_s,s)dB_s=\int_0^t \Big(f_s(B_s,s)+\frac{1}{2}f_{xx}(B_s,s)\Big)ds$$

is a continuous local martingale with finite variation, which is necessarily constant $\equiv 0$.

4. By using independence of increment, and the formula for the characteristic function of a standard gaussian $E(\exp(i\theta B_1))=\exp(-\theta^2/2)$, $i=\sqrt{-1}$, we have seen that

$$Z_t(\theta)=\exp\big(i\theta B_t+\frac{1}{2}\theta^2 t\big)=\cos(\theta B_t)\exp(\theta^2 t/2)+i\sin(\theta B_t)\exp(\theta^2 t/2)$$
$$=M_t(\theta)+iN_t(\theta)$$

is a complex valued $\mathbb{F}$-martingale, where

$$M_t(\theta)=\cos(\theta B_t)\exp(\theta^2 t/2),\quad N_t(\theta)=\sin(\theta B_t)\exp(\theta^2 t/2),$$

Equivalently M_t and N_t are real valued martingales.

- Check that M_t and N_t are in $L^2(P)$.
  Solution Since $|\cos(\theta)|\le 1$, $E(M_t^2)\le\exp(\theta^2)<\infty$
- Use Exercise 1 and 2 together with Ito formula to compute $\langle M(\theta)\rangle_t$, $\langle N(\theta)\rangle_t$, and $\langle N(\theta),M(\theta)\rangle_t$.
  Hint: express $M(\theta)_t=M(\theta)_0+\int_0^t f_x(s,B_s)dB_s$ as an Ito integral, use the formula $\langle Y\cdot M\rangle_t=\int_0^t Y_s^2 d\langle M\rangle_s$.
  Solution

$$dM_t=-N_t\theta dB_t,\quad dN_t=M_t\theta dB_t$$
$$M_t=1-\theta\int_0^t N_s dB_s,\quad N_t=\theta\int_0^t M_s dB_s,$$
$$\langle M\rangle_t=\theta^2\int_0^t N_s^2 ds,\quad \langle N\rangle_t=\theta^2\int_0^t M_s^2 ds,$$
$$\langle M,B\rangle_t=-\theta\int_0^t N_s ds,\quad \langle N,B\rangle_t=\theta\int_0^t M_s ds,\quad \langle M,N\rangle_t=-\theta^2\int_0^t M_sN_s ds$$

5. Compute $E(M_t^2(\theta))$ and $E(N_t^2(\theta))$

Hint: As an alternative to the direct calculation, use the isometry

$$E(M_t^2(\theta))=E(M_0(\theta)^2)+E(\langle M(\theta)\rangle_t),\quad E(N_t^2(\theta))=E(N_0(\theta)^2)+E(\langle N(\theta)\rangle_t)$$

and the previous exercise to show that

$$E(M_t^2(\theta))=1+\int_0^t E(N_s^2(\theta))ds,\quad E(N_t^2(\theta))=\int_0^t E(M_s^2(\theta))ds$$

which gives a deterministic 2-dimensional linear differential system with unknown functions $\xi_t = E(M_t^2(\theta)), \eta_t = E(N_t^2(\theta))$. To solve it use hyperbolic functions:

$$\sinh(x) = (e^x - e^{-x})/2, \quad \cosh(x) = (e^x + e^{-x})/2,$$

Solution By Fubini

$$E(M_2^t) = E(M_0^2) + E(\langle M_t \rangle) = 1 + \theta^2 \int_0^t E(N_s^2) ds,$$
$$E(N_2^t) = E(M_0^2) + E(\langle N_t \rangle) = 0 + \theta^2 \int_0^t E(M_s^2) ds,$$

i.e. $E(M_t^2) = \xi_t$ and $E(N_t^2) = \eta_t$ solving the linear differential system

$$\xi_t = 1 + \theta^2 \int_0^t \eta_s ds,$$
$$\eta_t = 0 + \theta^2 \int_0^t \xi_s ds,$$

In matrix form

$$d \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \theta^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} dt = \theta^2 A \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} dt$$

with solution

$$\begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \exp\big(\theta^2 A t\big) \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}$$

where $(\xi_0, \eta_0) = (1, 0)$, and we use the matrix exponential

$$\exp\big(\theta^2 A t\big) = \sum_{n=0}^{\infty} \frac{(\theta^2 t)^n}{n!} A^n (1,0)^\top$$

where $A^{2n} = \mathrm{Id}, \quad A^{2n+1} = A$.

$$\xi_t = \sum_{n=0}^{\infty} \frac{(\theta^2 t)^{2n}}{(2n)!} = \cosh(\theta^2 t) = \frac{1}{2}\big\{\exp(\theta^2 t) + \exp(-\theta^2 t)\big\}$$
$$\eta_t = \sum_{n=0}^{\infty} \frac{(\theta^2 t)^{2n+1}}{(2n+1)!} = \sinh(\theta^2 t) = \frac{1}{2}\big\{\exp(\theta^2 t) - \exp(-\theta^2 t)\big\}$$