Stochastic analysis, autumn 2011, Exercises-1, 13.09.11

1. A random vector $X = (X_1, \dots, X_n)$ is jointly gaussian if there is a vector $\mu \in \mathbb{R}^n$ and a $m \times n$ matrix A such that in matrix notation

$$X(\omega) = \mu + Y(\omega)A$$

where $Y = (Y_1, \dots, Y_m)$ are independent standard gaussian variables, with $E(Y_i) = 0$, $E(Y_i Y_j) = \delta_{ij}$.

- Compute the probability density of X.
- Compute the covariance $E(X_i X_j)$.

You can assume that $m = n$ and the matrix A is invertible.

Solutions

For a measurable test function $f(x_1, \dots, x_n)$,

$$E_P\big(f(X_1, \dots, X_n)\big) = E_P\big(f(\mu_1 + A_{1,\cdot} \cdot Y, \dots, \mu_1 + A_{1,\cdot})\big) =$$
$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\mu_1 + \sum_i A_{1i} y_i, \dots, \mu_d + \sum_i A_{di} y_i)$$
$$\times \exp(-\frac{1}{2} \sum_{i=1}^d y_i^2) dy_1 \dots dy_d$$

by the change of variable formula,

with $x=\mu+Ay$ with inverse $y=A^{-1}(x-\mu)$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}\sum_{i=1}^d \Big(\sum_{j=1}^d (A^{-1})_{ij}(x_j-\mu_j)\Big)^2\right) |J(x_1,\dots,x_d)| dx_1 dx_2 \dots dx_d$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}\sum_{i=1}^d \Big(\sum_{k=1}^d (A^{-1})_{ik}(x_k-\mu_k)\Big(\sum_{j=1}^d (A^{-1})_{ij}(x_j-\mu_j)\Big)^2\right) |\det(J(x_1,\dots,x_d))| dx_1 dx_2 \dots dx_d$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}\sum_{k=1}^d \sum_{j=1}^d \left(\sum_{i=1}^d (A^{-1})_{ik}(A^{-1})_{ij}\right)(x_k-\mu_k)(x_j-\mu_j) |\det(J(x_1,\dots,x_d))| dx_1 dx_2 \dots dx_d\right.$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}(x-\mu)\left(A^{-1}(A^{-1})^\top\right)(x-\mu)^\top\right) |J(x_1,\dots,x_d)| dx_1 dx_2 \dots dx_d$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}(x-\mu)\left(AA^\top\right)^{-1}(x-\mu)^\top\right) |\det(J(x_1,\dots,x_d))| dx_1 dx_2 \dots dx_d =$$

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x_1,\dots,x_d)$$

$$\times \exp\left(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^\top\right) |\det(J(x_1,\dots,x_d))| dx_1 dx_2 \dots dx_d$$

where $\Sigma = AA^\top$ is the covariance matrix of $(X_1,\dots,X_d)$ since

$$E((X_i-\mu_i)(X_j-\mu_j)) = E(A_{i\cdot}Y A_{j\,\mathrm{cot}}Y) = \sum_{k=1}^d\sum_{h=1}^d A_{ik}A_{jh}E(Y_kY_h) = \sum_{k=1}^d A_{ik}A_{jk} = (AA^\top)_{ij}$$

where $E(Y_kY_h)=\delta_{hk}$.

$J(x)$ is the Jacobian matrix of the linear transformation $x\mapsto y$

$$J_{ik}(x) = \left(\frac{\partial y_i}{\partial x_k}\right) = (A^{-1})_{hk}$$

which does not depend on $(x_1,\dots,x_d)$.

Note that $\det(A)=\det(A^\top)$ and $\det(AA^\top)=\det(A)\det(A^T)=\det(A)^2$.
Also $\det(A^{-1})=\det(A)^{-1}$. Therefore

$$|\det(A^{-1})| = |\det(A)^{-1}| = \det(AA^\top)^{-1/2} = \det(\Sigma)^{-1/2}$$

It follows that $X=(X_1,\dots,X_d)$ has density

$$p_X(x_1,\dots,x_d) = (2\pi)^{-d/2}\det(\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^\top\right) \quad (1)$$

2. Recall the definition: a standard Brownian motion $(B_t : t \geq 0)$ is a stochatic process with

   - $B_0 = 0$
   - the increments are indepenent gaussian with $(B_t - B_s) \sim \mathcal{N}(0, t-s)$, for $s \leq t$.
   - the trajectories are continuous.

   Show that the Brownian motion is a gaussian process, that is for all $n \in \mathbb{N}$ $0 \leq t_1 \leq \cdots \leq t_n$,

   $(B_{t_1}, \ldots, B_{t_n})$ is a jointly gaussian random vector.

   Hint: use the chain rule to write the joint density.

   Solution Instead we use the independence of increments: for $d \in \mathbb{N}$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_d$ denote $\Delta B_i = B_{t_i} - B_{t_{i-1}}$. Then

   $$(B_{t_1}, \ldots, B_{t_d}) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 1 & & 1 \end{pmatrix} \begin{pmatrix} \Delta B_1 \\ \vdots \\ \Delta B_d \end{pmatrix}$$

   where $(\Delta B_1, \ldots, \Delta B_d)$ are independent. $(B_{t_1}, \ldots, B_{t_d})$ is a jointly gaussian vector since it is the linear transformation of independent gaussian variables.

   We write the the joint density of $(B_{t_1}, \ldots, B_{t_d})$ using the Markov property (to be shown in one of the following exercises) and the chain rule:

   $$p(x_1, \ldots, x_d) =$$
   $$\frac{1}{\sqrt{t_1}} \phi\left(\frac{x_1}{\sqrt{t_1}}\right) \frac{1}{\sqrt{t_2 - t_1}} \phi\left(\frac{x_2 - x_1}{\sqrt{t_2 - t_1}}\right) \cdots \frac{1}{\sqrt{t_d - t_{d-1}}} \phi\left(\frac{x_d - x_{d-1}}{\sqrt{t_d - t_{d-1}}}\right)$$

   where $\phi(x)$ is the standard gaussian density.

   Alternatively compute first the covariance matrix: for $0 \leq s \leq t$

   $$E(B_s B_t) = E(B_s^2) + E(B_s(B_t - B_s)) = s + E(B_s)E(B_t - B_s) = s \wedge t$$

   by the independence of the increments. Therefore the joint density of $(B_{t_1}, \ldots, B_{t_d})$ is given by the expression (1) with $\mu = 0$ and $\Sigma_{ij} = t_i \wedge t_j$.

3. Let $(B_t : t \geq 0)$ a standard Brownian motion.

   For $t \in (0, 1)$ use Bayes formula to write the regular conditional density of B_t conditionally on B_1

   For $0 < t_1 < \cdots < t_n < 1$ compute the finite dimensional conditional distribution of $(B_{t_1}, \ldots, B_{t_n})$ given $\{B_1 = y\}$.

   For $s \leq t$ compute $E(B_t B_s | B_1 = 0)$

Solution By Bayes formula the conditional density is

$$p_{B_{t_1},\dots,B_{t_d}|B_1}(x_1,\dots,x_d|y) = \frac{p_{B_{t_1},\dots,B_{t_d},B_1}(x_1,\dots,x_d,y)}{p_{B_1}(y)} =$$
$$\frac{1}{\sqrt{t_1}}\phi\Big(\frac{x_1}{\sqrt{t_1}}\Big)\frac{1}{\sqrt{t_2-t_1}}\phi\Big(\frac{x_2-x_1}{\sqrt{t_2-t_1}}\Big)\dots\frac{1}{\sqrt{t_d-t_{d-1}}}\phi\Big(\frac{x_d-x_{d-1}}{\sqrt{t_d-t_{d-1}}}\Big)\frac{1}{\sqrt{1-t_d}}\phi\Big(\frac{y-x_d}{\sqrt{1-t_d}}\Big)\phi(y)^{-1} =$$
$$p_{B_{t_1}|B_1}(x_1,y)p_{B_{t_2}|B_{t_1},B_1}(x_2|x_1,y)\dots p_{B_{t_d}|B_{t_{d-1}}B_1}(x_d|x_{d-1},y)$$

where in the last expression we applied the chain rule to the conditional probability density and used Markov property. We comĂĕute the terms in the last product:

$$p_{B_{t_1}|B_1}(x_1,y) = \frac{1}{\sqrt{t_1}}\phi\Big(\frac{x_1}{\sqrt{t_1}}\Big)\frac{1}{\sqrt{1-t_1}}\phi\Big(\frac{y-x_1}{\sqrt{1-t_1}}\Big)\phi(y)^{-1}$$
$$\frac{1}{\sqrt{2\pi t_1(1-t_1)}}\exp\Big(-\frac{1}{2}\Big(\frac{x_1^2}{t_1}+\frac{(y-x_1)^2}{1-t_1}-y^2\Big)\Big) =$$
$$\frac{1}{\sqrt{2\pi t_1(1-t_1)}}\exp(-\frac{1}{2}\frac{x_1^2(1-t_1+t_1)+y^2t_1(1-1+t_1))-2yx_1t_1}{t_1(1-t_1)}\Big) =$$
$$\frac{1}{\sqrt{2\pi t_1(1-t_1)}}\exp\Big(-\frac{1}{2}\frac{(x_1-yt_1)^2}{t_1(1-t_1)}\Big)$$

which means B_{t_1} given $B_1=y$ is conditionally gaussian with conditional mean yt_1 and conditional variance $t_1(1-t_1)$ and

$$p_{B_{t_d}|B_{t_{d-1}},B_1}(x_d|x_{d-1},y) = p_{B_{t_d}-B_{t_{d-1}}|B_1-B_{t_{d-1}}}(x_d-x_{d-1}|y-x_{d-1}) =$$
$$\frac{1}{\sqrt{2\pi(t_d-t_{d-1})(1-t_d)}}\exp\Big(-\frac{1}{2}\frac{(x_d-x_{d-1}-(y-x_{d-1})(t_d-t_{d-1}))^2}{(t_d-t_{d-1})(1-t_d)}\Big)$$
$$=\frac{1}{\sqrt{2\pi(t_d-t_{d-1})(1-t_d)}}\exp\Big(-\frac{1}{2}\frac{(x_d-x_{d-1}-(y-x_{d-1})(t_d-t_{d-1}))^2}{(t_d-t_{d-1})(1-t_d)}\Big)$$
$$=\frac{1}{\sqrt{2\pi(t_d-t_{d-1})(1-t_d)}}\exp\Big(-\frac{1}{2}\frac{\Big(x_d-(x_{d-1}(1-t_d+t_{d-1})+y(t_d-t_{d-1}))\Big)^2}{(t_d-t_{d-1})(1-t_d)}\Big)$$

which means B_{t_d} given $B_{t_{d-1}}=x_{d-1}$ and $B_1=y$ is conditionally gaussian with conditional mean $\big(y(t_d-t_{d-1})+x_{d-1}(1-t_d+t_{d-1})\big)$ and conditional variance $(t_d-t_{d-1})(1-t_d)$.

4. When $f\in C^2$, from Ito Föllmer we get the *semimartingale decomposition* of the process $f(B_t(\omega))$ as an Ito integral plus a process with finite variation of on compacts. Write the semimartingale decomposition in the following cases $f(x)=x^n$; $f(x)=\sin(x)$; $f(x)=\exp(x)$.

$$B_t^n = n\int_0^t B_s^{n-1}dB_s+\frac{n(n-1)}{2}\int_0^t B_s^{n-2}ds$$
$$\sin(B_t) = \int_0^t\cos(B_s)dB_s-\frac{1}{2}\int_0^t\sin(B_s)ds$$
$$\exp(B_t) = \int_0^t\exp(B_s)dB_s+\frac{1}{2}\int_0^t\exp(B_s)ds$$

5. Use Ito formula to express $\exp(B_t - \frac{1}{2}t)$ as an Ito-Föllmer integral:

   Solution The process $X_t = B_t - \frac{1}{2}t$ has quadratic variation $[X]_t = [B]_t = t$, since the difference $(X_t - B_t) = \frac{1}{2}t$ has finite total variation and therefore zero quadratic variation. By Ito formula

$$\exp(B_t - \frac{1}{2}t) - 1 = \int_0^t \exp(B_s - \frac{1}{2}s)(dB_s - \frac{1}{2}ds) + \frac{1}{2}\int_0^t \exp(B_s - \frac{1}{2}s)ds$$
$$= \int_0^t \exp(B_s - \frac{1}{2}s)dB_s$$

   Alternatively we could use the multivariate Ito formula for $f(x,t) = \exp(x - \frac{1}{2}t)$

$$f(B_t,t) - 1 = \int_0^t \frac{\partial}{\partial x} f(B_s,s)dB_s + \int_0^t \left(\frac{\partial}{\partial t} f(B_s,s) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(B_s,s)\right)ds$$

   which gives the same result.

6. Let $(B_t(\omega))_{t\geq 0}$ and $(W_t(\omega))_{t\geq 0}$ two independent Brownian motions defined on the same probability space. Adapt the proof of lemma **??** to show that *quadratic covariation*

$$[W,B]_t = \lim_{\Delta(\Pi)\to 0} \sum_i (W_{t_{i+1}} - W_{t_i})(B_{t_{i+1}} - B_{t_i}) \stackrel{P}{\to} 0 \qquad (2)$$

   where we take the limit over partitions $\Pi = (0 = t_0 \leq t_1 \leq \cdots \leq t_n = t)$, $n \in \mathbb{N}$ as $\Delta(\Pi) \to 0$ Hint: take the limit in $L^2(P)$ and use independence.

   The process $[W,B]_t$ is called *quadratic covariation* .

   Solution

   We first show convergence in $L^2(P)$ (which implies convergence in probability) as $\Delta(\Pi_n)$. For the dyadic sequence of partitions $\Pi_n = D_n$ we have also almost sure convergence in (2).

   The since W and B are independent, the variance of the sum approximating sums is

$$E\Big(\Big\{\sum_{0<t_k^n\leq 1}(B_{t_{k+1}^n} - B_{t_k}^n)(W_{t_{k+1}^n} - W_{t_k}^n)\Big\}^2\Big) = \sum_{0<t_k^n\leq 1} E\big((B_{t_{k+1}^n} - B_{t_k^n})^2\big)E\big((W_{t_{k+1}^n} - W_{t_k^n})^2\big) = 2^n 2^{-2n}$$

   (since increments over disjoint intervals are independent the cross-product terms have zero expectation). This shows convergence in $L^2(P)$ and in probabiliy.

   Let $\varepsilon > 0$ and

$$A_n^\varepsilon = \big\{\omega : |\sum_{t_k^n\leq 1}(B_{t_{k+1}}^n(\omega) - B_{t_k}^n(\omega))(W_{t_{k+1}}^n(\omega) - W_{t_k}^n(\omega))| > \varepsilon\big\}$$

   by Chebychev inequality

$$P(A_n^\varepsilon) \leq 2^{-n}\varepsilon^{-2}$$

Therefore

$$\sum_n P(A_n^\varepsilon) \leq \varepsilon^{-2} \sum_{n=0}^{\infty} 2^{-n} < \infty$$

Applying Borel Cantelli lemma, $\forall \varepsilon > 0$

$$P\big(\limsup_n A_n^\varepsilon\big) = 0$$

Taking $\varepsilon = 1/m,\ m \in \mathbb{N}$ and countable intersection of the complements

$$P\Bigg(\bigcap_{m\geq 0} \bigcup_{k\geq 0} \bigcap_{n\geq k} A_n^{1/m}\Bigg) = 1$$

which is the probability that exists $[W, B]_1 = 0$ by taking limits among the dyadic sequence.