Stochastic analysis, autumn 2011, Exercises-9, 15.11.2011

1. (Martingale characterization) Let $M_t(\omega)$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $E(|M_t|) < \infty\ \forall t$. Then (X_t) is an $\mathbb{F}$-martingale if and only if $\forall$ $\mathbb{F}$-stopping times $\tau(\omega)$ which can take at most two finite values

$$E(M_\tau) = E(M_0)$$

   Hint: the necessity is just Doob optional sampling theorem, for sufficency for $s \le t$ and $A \in \mathcal{F}_s$ define $\tau(\omega) = t\mathbf{1}_A + s\mathbf{1}_{A^c}$ and show that it is a stopping time. Then use the hypothesis to show that $E\big((M_t - M_s)\mathbf{1}_A\big) = 0$.

2. (Lenglart's inequality)

   Let $X_t(\omega) \ge 0$ and $A_t(\omega) \ge 0$ continuous processes adapted with respect to $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $X_0 = 0$, and we assume that A_t is non-decreasing such that for all **bounded** stopping times $\tau(\omega)$

$$E(X_\tau) \le E(A_\tau)$$

   We introduce the running maximum $X_t^*(\omega) = \max\limits_{0 \le s \le t} X_s(\omega)$

   Prove the following inequalities for **all** $\mathbb{F}$-stopping times τ, (also unbounded): $\forall \varepsilon, \delta > 0$

$$\textbf{a)} \qquad P(X_\tau^* > \varepsilon) \le \frac{E(A_\tau)}{\varepsilon}$$

$$\textbf{b)} \qquad P(X_\tau^* > \varepsilon, A_\tau \le \delta) \le \frac{E(A_\tau \wedge \delta)}{\varepsilon}$$

$$\textbf{c)} \qquad P(X_\tau^* > \varepsilon) \le \frac{E(\delta \wedge A_\tau)}{\varepsilon} + P(A_\tau > \delta)$$

   Hint: show it first for bounded stopping times, then use monotone convergence.

3. Let M_t a continous $\mathbb{F}$-local martingale. The $\mathbb{F}$-predictable variation $\langle M \rangle_t$ is the non-decreasing process with $\langle M \rangle_0 = 0$ such that

$$M_t^2 - \langle M \rangle_t$$

   is a $\mathbb{F}$-local martingale.

   Show that for any $\mathbb{F}$Âăstopping time τ

$$P\left(\max_{0 \le s \le t} |M_s(\omega)| > \varepsilon\right) \le \frac{E(\delta \wedge \langle M \rangle_\tau)}{\varepsilon^2} + P(\langle M \rangle_\tau > \delta)$$

4. $\{M_t^{(n)}(\omega)\}_{n \in \mathbb{N}}$ a sequence of $\mathbb{F}$-local martingales and τ a $\mathbb{F}$-stopping time. Show that

$$\langle M^{(n)} \rangle_\tau \xrightarrow{P} 0 \quad \Longrightarrow \quad \max_{0 \le s \le \tau} |M_s(\omega)| \xrightarrow{P} 0$$

   where we use convergence in probability.

5. (A discontinous martingale) Consider a random time $\tau(\omega) \in [0, \infty]$ with distribution function $F(t) = P(\tau \leq t)$.

Define the Riemann Stieltjes integrals

$$\Lambda_t = \int_0^t \frac{1}{1 - F(s-)} F(ds) = \int_0^t \frac{1}{P(\tau \geq s)} F(ds) = \int_0^t P(\tau \in ds | \tau \geq s), \quad t \geq 0$$

For simplicity you can assume that F is continuous or even absolutely continuous.

Consider counting process $N_t(\omega) := \mathbf{1}(\tau(\omega) \leq t)$ which has one jump at size 1 at time τ, And let $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$. with $\mathcal{F}_t = \sigma(N_s : s \leq t)$.

- Show that τ is $\mathbb{F}$-stopping time.
- Show that $M_t := N_t(\omega) - \Lambda(t \wedge \tau(\omega))$ is a $\mathbb{F}$-martingale.

  Hint: show that $\mathcal{F}_s = \Big\{ \{\tau(\omega) > s\}$and $\{\tau(\omega) \in B\} : B \subset [0, s]$ Borel $\Big\}$

  For such sets $A \in \mathcal{F}_s$ and $s \leq t$ compute $E((M_t - M_s)\mathbf{1}_A)$.