Stochastic analysis, autumn 2011, Exercises-8, 08.11.2011

1.
   - Let $\tau(\omega) \in \mathbb{R}^+ \cup \{+\infty\}$ a stopping time w.r.t. to a filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$.

     Show that for all $\varepsilon \geq 0$, $(\tau(\omega) + \varepsilon)$ is also a $\mathbb{F}$-stopping time.
   - Suppose that $\tau(\omega) \in \mathbb{R}^+ \cup \{+\infty\}$ a $\mathbb{F}$-stopping time and for some $\varepsilon > 0$, $\tau_{-\varepsilon}(\omega) := (\tau(\omega) - \varepsilon) \vee 0$ is a $\mathbb{F}$-stopping time as well.

     What can you say about τ?
   - Construct a stopping time $\tau(\omega)$ such that for all $\varepsilon > 0$, $\tau_{-\varepsilon}(\omega) := (\tau(\omega) - \varepsilon) \vee 0$ is not a stopping time. Hint: consider the first hitting time at some level a in the the filtration generated by a Brownian motion (B_t), with $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$.

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and $\mathbb{G}$ a filtration with $\mathcal{G}_t \subseteq \mathcal{F}_t$ We assume that (M_t) is also $\mathbb{G}$-adapted. Show that (M_t) is a martingale in the smaller filtration $\mathbb{G}$.

3. Let $(M_t : t \in \mathbb{R})$ a F-martingale under P, and $\mathcal{G}_t$ a filtration such that $\forall t \geq 0$, the σ-algebrae $\mathcal{G}_t$ and $\sigma(M_s : s \leq t)$ are P-independent.

   Show that under P, $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\mathcal{F}_t \vee \mathcal{G}_t : t \geq 0)$.

4. Let $(B_t : t \geq 0)$ a Brownian motion in the filtration $\mathbb{F}$, which means
   - $B_0(\omega) = 0$
   - $t \mapsto B_t(\omega)$ is continuous
   - $\forall 0 \leq s \leq t$, $(B_t - B_s)$ is P-independent from $\mathcal{F}_s$, conditionally gaussian with conditional mean $E(B_t - B_s|\mathcal{F}_s) = 0$ and conditional variance $E((B_t - B_s)^2|\mathcal{F}_s) = t - s$

   Let $\theta \in \mathbb{R}$, $\theta \neq 0$.

   - Show that $Z_t = \exp(\theta B_t - \frac{1}{2}\theta^2 t)$ is a $\mathbb{F}$-martingale.

     Hint: use independence of increments and remember that $E(\exp(\theta G)) = \exp(\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0, \sigma^2)$.
   - Show that the limit $\lim_{t\to\infty} Z_t(\omega)$ exists P almost surely. Hint: use Doob martingale convergence theorem.
   - Show that $(Z_t : t \geq 0)$ is not uniformly integrable. Hint: use Kakutani's theorem
   - Show that $\lim_{t\to\infty} Z_t(\omega) = 0$, P- almost surely.

5. For $\theta \in \mathbb{R}$, consider now $M_t = \exp(i\theta B_t + \frac{1}{2}\theta^2 t) \in \mathbb{C}$ where $i = \sqrt{-1}$ is the imaginary unit.

   Recall that $E(\exp(i\theta G)) = \exp(-\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0, \sigma^2)$.

   - Show that M_t is complex valued $\mathbb{F}$-martingale, which means that real and imaginary parts are $\mathbb{F}$-martingales.
   - Show that $\lim_{t\to\infty} M_t(\omega) = \infty$.