Stochastic analysis, autumn 2011, Exercises-7, 01.11.2011

1. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

   We introduce the random variables

   $X_t(\omega) = \mathbf{1}\big\{$ the ball drawn at time t is black $\big\}$

   and denote $S_t = (1 + X_1 + \dots + X_t)$,

   $M_t = S_t/(t + 2)$, the proportion of black balls in the urn.

   We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : s \in \mathbb{N}, s \le t\}$.

   i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

   ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

   iv) Note that the martingale $(M_t)_{t \ge 0}$ is uniformly integrable (Why ?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

   v) Show that $P(0 < M_\infty < 1) > 0$.

   Since $M_\infty(\omega) \in [0, 1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

   Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.

2. A branching process $(Z_t)_{t\in\mathbb{N}}$ with integer values, represents the size of a population evolving randomly in discrete time.

   We start with $Z_0(\omega) = 1$ individual at time $t = 0$.

   Inductively each of the $Z_{t-1}(\omega)$ individuals in the $(t - 1)$ generation has a random number of offspring $X_{i,t}$. These offspring numbers are independent and identically distributed with law $\pi = (\pi(n) : n = 0, 1, \dots)$,

   $\pi(n) = P\big(X_{i,t} = n\big)$.

   The size of the new generation at time t is then

   $$Z_t(\omega) = \sum_{i=1}^{Z_{t-1}(\omega)} X_{i,t}(\omega)$$

We assume that the mean offspring number is finite

$$\mu = E_\pi(X) = \sum_{n=0}^{\infty} n\pi(n) < \infty$$

- Show that $Z_t(\omega)$ is a martingale, (respectively supermartingale, submartingale) when $\mu = 1$ (respectively $0 \leq \mu < 1$, $1 < \mu < \infty$, in the filtration generated by the process Z itself.
- For $\mu \neq 1$, write the Doob decomposition of Z_t and compute the mean $E(Z_t)$ for $t \in \mathbb{N}$.
- Assume that $\mu \leq 1$, and that the offspring distribution is non-trivial, meaning that $0 \leq \pi(X = 1) < 1$. The case $P(X = 1) = 1$ is trivial, nothing happens.

  Show that

  $$\lim_{t\to\infty} Z_t(\omega) = 0 \qquad P \text{ a.s.}$$

  Hint: first show that a finite limit $Z_\infty(\omega)$ exists P a.s. Consider

  $$P(Z_\infty = 0|Z_1 = n) = P(Z_\infty = 0)^n$$

  since $P(Z_\infty = 0)$ is the probability that descendance of a single individual becomes extinct, is the probability that independently for each of its children the respective descendances become extinct.

  By computing first the conditional probability $P(Z_\infty = 0|\sigma(Z_1))(\omega)$ and taking expectation, show that the unknown $q = P(Z_\infty = 0)$ satisfies the equation

  $$q = E_P(q^X), \quad q \in [\pi(0), 1]$$

  where $P(X = n) = \pi(n)$ is the offspring distribution.

  Note that since $\mu = E(X) \leq 1$ and $\pi(1) = P(X = 1) < 1$, necessarily $\pi(0) = P(X = 0) > 0$, and $P(Z_\infty = 0) \geq P(X = 0) > 0$. Therefore the $q = 0$ is not a solution.

  $q = 1$ is also a solution. We show that there are no other solutions. Note that by Jensen inequality for the concave function $x \mapsto q^x$ with $q \in [0, 1]$

  $$E(q^X) \geq q^{E(X)} \geq q$$

  Show that the inequality is strict in the non-trivial case:

  If $0 < q < 1$ cannot be a solution since the derivative

  $$\frac{d}{dq} E_P(q^X) = E\left(\frac{d}{dq} q^X\right) = E(Xq^{X-1}) < E(X) \leq 1$$

  with strict inequality in the non-trivial case in the $P(X = 1) < 1$.

You need to check that it is allowed to take a derivative inside the expectation.
This implies that

$$E_P(q^X) > q, \forall \quad q \in (0,1)$$

- Assume that $\mu = 1$. Show that the martingale $(Z_t : t \in \mathbb{N})$ is not uniformly integrable

3. We now make a change of measure and define a new measure P' such that under Q the offspring numbers are independent and identically distributed with

$$P'(X_{i,t} = n) = \pi'(n)$$

with $\pi'(n) = 0$ when $\pi(n) = 0$. Compute the likelihood ratio process $\frac{dP'_t}{dP_t}(\omega)$ on the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(X_{i,s} : i \in \mathbb{N}, 1 \le s \le t)$.

4. Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0,1]$, $P(U_1 \in dx) = \mathbf{1}_{[0,1]}(x)dx$. Note that $E_P(U_t) = 1/2$.

Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

$$P(-\log(U_1) > x) = \begin{cases} \exp(-x) & \text{kun } x \ge 0 \\ 1 & \text{kun } x < 0 \end{cases}$$

$-\log(U_1) \in L^1(P)$ with $E_P(-\log(U_1)) = 1$.

- Let $Z_0 = 1$, and

$$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, Z_2, \dots, Z_t) = \sigma(U_1, U_2, \dots, U_t)$.
- Show that $E_P(Z_t) = 1$.
- Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.
- Show that

$$Z_\infty(\omega) = 0 \qquad P\text{-a.s.}$$

Hint Compute first the P-a.s. limit

$$\lim_{t\to\infty} \frac{1}{t} \log(Z_t(\omega))$$

(remember Kolmogorov's strong law of large numbers!).
- Show that the martingale $(Z_t(\omega) : t \in \mathbb{N})$ is not uniformly integrable.
- Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem ?
- At every time $t \in \mathbb{N}$, define the probability measure

$$Q_t(A) := E_P(Z_t \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$

on the probability space $(\Omega, \mathcal{F})$.
Show that the random variables $(U_1, \dots, U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.