Stochastic analysis, autumn 2011, Exercises-4, 4.10.11

1. Show:
   - For $X, Y \in L^2(P)$,
     $$\| X+Y \|_2^2 + \| X-Y \|_2^2 = 2 \| X \|_2^2 + 2 \| Y \|_2^2 \qquad \text{(parallelogram identity)}$$
   - $$E_P(XY) = \frac{1}{4}\big(\| X+Y \|_2^2 - \| X-Y \|_2^2\big) \qquad \text{(Polarization identitity)}$$
   - A norm $\| x \|$ satisifies the parallelogram identity if and only if the polarization identitety defined a scalar product (x, y) (bilinear, symmetric and positive) such that $\| x \|^2 = (x, x)$.

2. Consider random variable $X(\omega)$ on $(\Omega, \mathcal{F}, P)$, assume either
   - **a)** $P(X \in dx) = \mathbf{1}(x \geq 0)\exp(-x)dx$, X on 1-eskponentiaalinen

     or
   - **b)** $P(X \in dx) = \pi^{-1}(1+x^2)^{-1}dx$, X on Cauchy jakautunut.

   Show that under **a)** $E_P(|X|) < \infty$ while under **b)** $E_P(|X|) = \infty$

   Let $Y(\omega) = \lfloor X(\omega) \rfloor = \max\{n \in \mathbb{Z} : n \leq X(\omega)\} \in \mathbb{Z}$.

   Show under **a**) and under **b**) the conditional expectation
   $$E_P(X|\sigma(Y))(\omega) \in \mathbb{R}$$
   Hint Note that theσ-algebra $\sigma(Y)$ is countably generated, by a countable $\mathcal{F}$-measurable partition of Ω. In this case the values taken by the conditional expectation (which is a random variable) are elementary conditional expectations obtained by conditioning on events of positive probability.

3. Let $X(\omega)$ and $Y(\omega)$ P-independent and identically distributed on $[0, 1]$:
   $$P(X \in dx, Y \in dy) = \mathbf{1}_{[0,1]}(x)\ \mathbf{1}_{[0,1]}(y)\ dx\ dy$$
   Let $Z(\omega) = \min(X(\omega), Y(\omega))$ Compute the conditional expectation $E_P(X|\sigma(Z))(\omega)$.

   Vihje It may be easier take first conditional expectation with respect to a larger σ-algebra:
   $$E_P(Z|\sigma(Z)) = E_P\big(E_P(X|\sigma(Z, I))\big|\sigma(Z)\big)$$
   with $I(\omega) := \mathbf{1}(X(\omega) \leq Y(\omega))$ and $\sigma(Z, I) \supseteq \sigma(Z)$.

4. Using the definition of conditional expectation together with Fatou's lemma for expectations,

   prove Fatou's lemma for conditional expectations: if $0 \leq X_n(\omega)$, $\forall n \in \mathbb{N}$
   $$0 \leq E_P(\liminf X_n|\mathcal{G})(\omega) = \liminf_n E_P(X_n|\mathcal{G})(\omega)$$

5. Let $X, Y \in L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra Check the identity
   $$\mathrm{Cov}_P(X, Y) = E_P\big(\ \mathrm{Cov}(X, Y|\mathcal{G})\big) + \mathrm{Cov}_P\big(E_P(X|\mathcal{G}), E_P(Y|\mathcal{G})\big)$$
   when $\mathcal{G} = \sigma(X)$ or $\mathcal{G} = \sigma(Y)$.