Stochastic analysis, autumn 2011, Exercises-3, 27.09.11

1. - Show that the linear space generated by the Haar system, which coincides with the set of functions which are piecewise constant on the dyadic partition D_n for some $n \in \mathbb{N}$, is dense in the space of continuous functions $C([0,1],\mathbb{R})$ under the supremum norm

$$\| f \|_\infty := \sup_{t\in[0,1]} |f(t)|$$

   We recall Luzin's theorem from real analysis: if $x : [0,1] \to \mathbb{R}$ is a measurable function, for all $\varepsilon > 0$ there exists a continuous function f such that

$$\lambda(\{t : x(t) \neq f(t)\}) < \varepsilon$$

   where $\lambda(dt)$ is Lebesgue measure.

   - Show that $C([0,1],\mathbb{R})$ is dense in $L^2([0,1],dt)$.
   - Show that the Haar system is a complete orthonormal basis of $L^2([0,1],dt)$.

2. Let $X(\omega) \in \mathbb{R}$ and $Y(\omega) = (Y_1(\omega),\dots,Y_d(\omega)) \in \mathbb{R}^d$ with $X, Y_i \in L^2(\Omega,\mathcal{F},P)$.

   Consider the linear subspace generated by $Y(\omega)$

$$\text{Lin}(Y) = \{a + b\cdot Y(\omega) : a \in \mathbb{R}, b \in \mathbb{R}^d\}.$$

   Note that this is a $(d+1)$-dimensional space.

   We define the best linear estimator of X given Y as the L^2-orthogonal projection $\hat{E}(X|Y)$ of X on the linear subspace $\text{Lin}(Y)$ generated by Y.

   Equivalently

$$\hat{E}(X|Y)(\omega) = \hat{a} + \hat{b}\cdot Y(\omega)$$

   for some deterministic $\hat{a} \in \mathbb{R}$ $\hat{b} \in \mathbb{R}^d$ where

$$(\hat{a}, \hat{b}Y(\omega)) = \arg\min_{a,b} E\big(\big\{X - (a + b\cdot Y)\big\}^2\big)$$

   Note that the conditional expectation $E(X|Y) = E(X|\sigma(Y))$ is the L^2-orthogonal projection of X on the infinite dimensional subspace $L^2(\Omega,\sigma(Y),P) \supset \text{Lin}(Y)$, and in general $E(X|Y) \neq \hat{E}(X|Y)$.

   - Show that

$$\hat{E}(X|Y) = E(X) + (Y - E(Y))\Sigma_{YY}^{-1}\Sigma'_{XY}$$
$$E\Big((X - \hat{E}(X|Y))^2\Big) = \Sigma_{XX}\Sigma_{YY}^{-1}\Sigma'_{XY}$$

   where the covariance matrix of $(X,Y) = (X, Y_1,\dots,Y_d)$ is denoted as

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma'_{XY} & \Sigma_{YY} \end{pmatrix}$$

   Hint: assume $E(X) = E(Y_i) = 0$, and maximize the mean square error with respect to the parameters a, b.

- Show that when the vector (X,Y) is jointly gaussian, all conditional distributions are gaussian and the best linear estimator $\hat{E}(X|Y)$ coincides with the conditional expectation $E(X|Y)$. (Use Bayes formula!).
  Hint: recall that the joint distribution of a gaussian vector is specified by the mean vector and covariance matrix.

3. Let $(B_t(\omega) : t \in [0,1])$ a Brownian motion, and $D_n = (k2^{-n} : k = 0,1,\dots,2^n)$.

   Show that for fixed n and dyadic indexes

   $$d = (2k+1)2^{-n} \in D_n \setminus D_{n-1}, d_- = 2k2^{-n}, d_+ = (2k+2)2^{-n} \in D_{n-1}$$

   with $k = 0,\dots 2^{n-1}$,

   $$G_d(\omega) := \biggl(B_d(\omega) - \frac{B_{d-}(\omega) + B_{d+}(\omega)}{2} \biggr) 2^{(n+1)/2}, \quad ,d \in D$$

   are i.i.d. standard gaussian variable $(E(G_d) = 0, E(G_d^2) = 0)$.

4. Let $G(\omega) \sim \mathcal{N}(0,1)$, and $f \in L^2(\mathbb{R}, d\gamma)$ where $\gamma(dx) = \phi(x)dx$.

   Here

   $$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-\frac{x^2}{2}\bigr)$$

   denotes the standard gaussian density.

   Consider the function

   $$u(t,x) = E_P\Bigl(f(x + G\sqrt{t}) \Bigr)$$

   - Show that $u(t,x)$ is smooth in the open set $(0,\infty) \times \mathbb{R}$. This does not require any smoothness on f.
     Hint: write

     $$\frac{u(t+\varepsilon,x) - u(t,x)}{\varepsilon}, \quad \frac{u(t,x+\varepsilon) - u(t,x)}{\varepsilon}$$

     as integrals, and do an opportune change of variable in order to use the smoothness of the gaussian density ϕ when you take the limit as $\varepsilon \to 0$.
   - Use the gaussian integration by parts formula to express the partial derivatives for $t > 0$

     $$\frac{\partial}{\partial t} u(t,x), \quad \frac{\partial}{\partial x} u(t,x), \quad \frac{\partial^2}{\partial x^2} u(t,x)$$

5. Let

   $$p(x,t) = \frac{1}{\sqrt{t}} \phi\bigl(\frac{x}{\sqrt{t}}\bigr)$$

- By using the Markov property of Bronwian motion (which follows from the independence of increments), show that for $0 \leq t \leq T$

$$p(y-x, T-t)dy = P(B_T \in dy | B_t = x) = P(B_{T-t} + x \in dy)$$

Denote for $t \in [0, T]$

$$v(t,x) = \int_{\mathbb{R}} g(y) p(y-x, T-t) dy = E_P(f(B_T) | B_t = x)$$

for some $f \in L^2(\mathbb{R}, d\gamma)$, where $\gamma(dy) = p(y, T)dy$ is the $\mathcal{N}(0, T)$ gaussian measure.

- Show that $v(t,x)$ is smooth in $[0,T) \times \mathbb{R}$ with respect to the variables (t,x), the partial derivatives

$$\frac{\partial}{\partial t} v(t,x), \quad \frac{\partial}{\partial x} v(t,x), \quad \frac{\partial^2}{\partial x^2} v(t,x)$$

- Show that $v(t,x)$ satifies the partial differential equation (heath equation)

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} v(t,x) \quad , 0 \leq t < T$$