**Stochastic analysis, autumn 2011, Exercises-2, 23.09.11**

1. Let

$$f(x) = f(0) + \int_0^x \dot{f}(y)dy, \quad h(x) = h(0) + \int_0^x \dot{h}(y)dy,$$

absolutely continous function with $\dot{f}, \dot{h} \in L^2(\mathbb{R}, \phi(x)dx)$ where $\phi(x)dx$ is the standard gaussian measure.

- For a standard gaussian random variable $G(\omega)$ with $E(G) = 0,\ E(G) = 1$ prove the gaussian integration by parts formula:

$$E_P\Big( f'(G)h(G) \Big) = E_P\Big( f(G)(Gh(G) - h'(G)) \Big)$$

Hint: rewrite the expectation as integral, and use integration by parts with respect to Lebesgue measure.

Assume first that $f(x)$ is compactly supported, and note that one can approximate $f(x)$ in $L^2(\mathbb{R}, \phi(x)dx)$ by absolutely continuous compactly supported functions.

- Write the corresponding gaussian integration by parts for $B_t(\omega) \sim \mathcal{N}(0, T)$ with $T > 0$

2. Show that a process $X_t$ with independent increments is Markov, which means for $0 \le s \le t$ and$\mathcal{F}_s^X = \sigma(X_u : 0 \le u \le s)$ and a bounded measurable test function $f(x)$

$$E(f(X_t)|\mathcal{F}_s^X) = E(f(X_t)|\sigma(X_s))$$

Hint. use first the decomposition $X_t = X_s + (X_t - X_s)$ and the definition of conditional expectation to show first that for a bounded measurable test-function

$$E(f(X_t)|\sigma(X_s))(\omega) = E\Big( f(y + (X_t - X_s)) \Big)\Big|_{y = X_s(\omega)}$$

3. $(N_t(\omega) : t \ge 0)$ is a $\lambda$-Poisson process ($\lambda > 0$) if $N_0 = 0$ and it has independent increments with $(N_t - N_s) \sim \text{Poisson}(\lambda(t - s))$

- Show that the Poisson process is non-decreasing with piecewise constant trajectories, it is $P$-almost surely finite on and increases only by jumps of size 1.
Hint: Show that

$$P\Big( \exists t \le T : \Delta N_t \ge 2 \Big) = 0$$

Write it as limits of probabilities of events depending on a finite number of increments.

- Compute the probability density of the first jump time $\tau(\omega)$.
Hint: $P(\tau > t) = P(N_t = 0)$.

4.

5. For a fixed $t \in [0,1]$ consider the function $s \mapsto h_t(s) := (s \wedge t) = \min(s,t)$

- Show that $h_t(\cdot) = (t \wedge \cdot)$ belongs to the Cameron-Martin space $H$.
- Show that $B_t(\omega) := B(h_t)$ is a Brownian motion.
- Show that $K(t,s) := \mathrm{Cov}(B_t, B_s) = E(B_t B_s) = s \wedge t$.
- Show the reproducing kernel Hilbert space property in the Cameron Martin space $H$: for $h \in H$

$$\big(K(t,\cdot), h(\cdot)\big)_H = h(t)$$