Stochastic analysis, autumn 2011, Exercises-10, 22.11.2011

1. Let τ be a $\mathbb{F}$-stopping time in the filtration generated by a Brownian motion B_t, such that $E(\tau) < \infty$.

   - Use Doob maximal inequality to show that $(B_{\tau\wedge t} : t \in \mathbb{R}^+)$ is a martingale bounded in $L^2(P)$.
   - Prove Wald's identities

   $$E(B_\tau) = 0 \quad , \quad E(B_\tau^2) = E(\tau)$$

   Hint: Doob optional sampling theorem cannot be applied directly since $(B_t : t \in \mathbb{R}^+)$ is not uniformly integrable, neither $\tau(\omega)$ is assumed to be bounded. Note also that

   $$B_\tau(\omega) = \sum_{n=1}^{\infty} \big(B_{\tau\wedge n}(\omega) - B_{\tau\wedge(n-1)}(\omega) \big)$$

2. Let M_t a continuous $\mathbb{F}$-martingale with $E(M_t^2) < \infty \ \forall t$, and let A_t be a continuous and bounded $\mathbb{F}$-adapted process with finite variation on finite intervals.

   Show that for $0 \le s \le t$

   $$M_t A_t - M_s A_s = \int_0^t A_s dM_s + \int_0^t M_s dA_s$$

   where on the right side we have an Ito integral and a Riemann Stieltjes integral.

   Note that the Ito integral $((A \cdot M)_t : t \in \mathbb{R}^+)$ is a square integrable martingale (why ?).

   Hint Note that

   $$M_t A_t - M_s A_s = M_t (A_t - A_s) + A_s (M_t - M_s)$$

   and use telescopic sums for some $s = r_0 < r_1 < \cdots < r_n = t$, letting the step-size of the partition going to zero.

3. Let B_t a Brownian motion and denote by $\mathbb{F}$ its filtration.

   Consider the pathwise Ito-Föllmer formula.

   $$f(B_t, t) = f(B_0, 0) + \int_0^t f_x(B_s, s) dB_s + \int_0^t \left(f_s(B_s, s) + \frac{1}{2} f_{xx}(B_s, s) \right) ds$$

   where the pathwise Föllmer integral coincides with the Ito integral. We assume that $f(x, s)$ is such that the the integrals above exist. Since the gaussian distribution has exponential moments, it is more than enough to assume that the derivatives have polynomial growth.

   Show that if $f(B_t, t)$ is a local martingale, necessarily

   $$f(B_t, t) = f(B_0, 0) + \int_0^t f_x(B_s, s) dB_s$$

   Hint: a local martingale with finite variation is constant.

4. By using independence of increment, and the formula for the characteristic function of a standard gaussian $E(\exp(i\theta B_1)) = \exp(-\theta^2/2)$, $i = \sqrt{-1}$, we have seen that

$$Z_t(\theta) = \exp\big(i\theta B_t + \frac{1}{2}\theta^2 t\big) = \cos(\theta B_t)\exp(\theta^2 t/2) + i\sin(\theta B_t)\exp(\theta^2 t/2)$$
$$= M_t(\theta) + iN_t(\theta)$$

is a complex valued $\mathbb{F}$-martingale, where

$$M_t(\theta) = \cos(\theta B_t)\exp(\theta^2 t/2), \quad N_t(\theta) = \sin(\theta B_t)\exp(\theta^2 t/2),$$

Equivalently M_t and N_t are real valued martingales.

- Check that M_t and N_t are in $L^2(P)$.
- Use Exercise 1 and 2 together with Ito formula to compute $\langle M(\theta)\rangle_t$, $\langle N(\theta)\rangle_t$, and $\langle N(\theta), M(\theta)\rangle_t$.

  Hint: express $M(\theta)_t = M(\theta)_0 + \int\limits_0^t f_x(s, B_s)dB_s$ as an Ito integral, use the formula $\langle Y \cdot M\rangle_t = \int_0^t Y_s^2 d\langle M\rangle_s$.

5. Compute $E(M_t^2(\theta))$ and $E(N_t^2(\theta))$

   Hint: As an alternative to the direct calculation, use the isometry

$$E(M_t^2(\theta)) = E(M_0(\theta)^2) + E(\langle M(\theta)\rangle_t), \quad E(N_t^2(\theta)) = E(N_0(\theta)^2) + E(\langle N(\theta)\rangle_t)$$

   and the previous exercise to show that

$$E(M_t^2(\theta)) = 1 + \int_0^t E(N_s^2(\theta))ds, \quad E(N_t^2(\theta)) = \int_0^t E(M_s^2(\theta))ds$$

   which gives a deterministic 2-dimensional linear differential system with unknown functions $\xi_t = E(M_t^2(\theta)), \eta_t = E(N_t^2(\theta))$. To solve it use hyperbolic functions:

$$\sinh(x) = (e^x - e^{-x})/2, \quad \cosh(x) = (e^x + e^{-x})/2,$$