Stochastic analysis, autumn 2011, Exercises-1, 13.09.11

1. A random vector $X = (X_1, \dots, X_n)$ is jointly gaussian if there is a vector $\mu \in \mathbb{R}^n$ and a $m \times n$ matrix A such that in matrix notation
$$X(\omega) = \mu + Y(\omega)A$$
where $Y = (Y_1, \dots, Y_m)$ are independent standard gaussian variables, with $E(Y_i) = 0$, $E(Y_i Y_j) = \delta_{ij}$.
   - Compute the probability density of X.
   - Compute the covariance $E(X_i X_j)$.

   You can assume that $m = n$ and the matrix A is invertible.

2. Recall the definition: a standard Brownian motion $(B_t : t \geq 0)$ is a stochatic process with
   - $B_0 = 0$
   - the increments are indepenent gaussian with $(B_t - B_s) \sim \mathcal{N}(0, t-s)$, for $s \leq t$.
   - the trajectories are continuous.

   Show that the Brownian motion is a gaussian process, that is for all $n \in \mathbb{N}$
   $0 \leq t_1 \leq \cdots \leq t_n$,
   $(B_{t_1}, \dots, B_{t_n})$ is a jointly gaussian random vector.
   Hint: use the chain rule to write the joint density.

3. Let $(B_t : t \geq 0)$ a standard Brownian motion.
   For $t \in (0,1)$ use Bayes formula to write the regular conditional density of B_t conditionally on B_1
   For $0 < t_1 < \cdots < t_n < 1$ compute the finite dimensional conditional distribution of $(B_{t_1}, \dots, B_{t_n})$ given $\{B_1 = 0\}$.
   For $s \leq t$ compute $E(B_t B_s | B_1 = 0)$

4. When $f \in C^2$, from Ito Föllmer we get the *semimartingale decomposition* of the process $f(B_t(\omega))$ as an Ito integral plus a process with finite variation of on compacts. Write the semimartingale decomposition in the following cases $f(x) = x^n$; $f(x) = \sin(x)$; $f(x) = \exp(x)$.

5. Use Ito formula to express $\exp(B_t - \frac{1}{2}t)$ as an Ito-Föllmer integral

6. Let $(B_t(\omega))_{t \geq 0}$ and $(W_t(\omega))_{t \geq 0}$ two indepenent Brownian motions defined on the same probability space. Adapt the proof of lemma **??** to show that *quadratic covariation*
$$[W, B]_t = \lim_{\Delta(\Pi) \to 0} \sum_i (W_{t_{i+1}} - W_{t_i})(B_{t_{i+1}} - B_{t_i}) \overset{P}{\to} 0 \qquad (1)$$
where we take the limit over partitions $\Pi = (0 = t_0 \leq t_1 \leq \cdots \leq t_n = t)$, $n \in \mathbb{N}$ as $\Delta(\Pi) \to 0$ Hint: take the limit in $L^2(P)$ and use independence.
   The process $[W, B]_t$ is called *quadratic covariation* .
   Show that for the dyadic sequence of partitions $\Pi_n = D_n$ we have also almost sure convergence in (1).