1. Prove Proposition 8.1 in the lecture note.

Hint: To prove that the solutions stay in $\mathcal{S}$: (i) Show that $I \geq 0$. (ii) Show that $\dot{N} - \dot{S} - \dot{I} = \gamma I - \mu(N - S - I) \geq -\mu(N - S - I)$, and so $N \geq S + I$. (iii) Show that $\dot{S} \geq (\beta - \mu - \sigma I)S$, and so $S \geq 0$.

2.[Simple SI model with demographics] Consider the following model.

$$\frac{dS}{dt} = B - \mu S - \sigma SI, \tag{1}$$

$$\frac{dI}{dt} = \sigma SI - (\mu + \alpha)I. \tag{2}$$

Here we assume that there is no recovery from the diseases, and there is a constant stream of new susceptibles, with rate B, coming into the system. All other parameters have the same meaning as the model in Section 8.1, and all parameters are assume to be strictly positive. The state-space is $\mathcal{S} = \{(S, I) \in \mathbb{R}^2 : S \geq 0,\ I \geq 0\}$.

- Show that $(S^o, I^o) = \big(\frac{B}{\mu}, 0\big)$ and $(S^*, I^*) = \big(\frac{\mu+\alpha}{\sigma}, \frac{\mu}{\sigma}(\frac{B}{\mu}\frac{\sigma}{\mu+\alpha} - 1)\big)$ are equilibria of the model, and that there is no other equilibrium.

  Note that the equilibrium (S^*, I^*) is only meaningful if $\frac{B}{\mu}\frac{\sigma}{\mu+\alpha} \geq 1$, and that if $\frac{B}{\mu}\frac{\sigma}{\mu+\alpha} = 1$, then $(S^*, I^*) = (S^o, I^o)$.

- Show that if $\frac{B}{\mu}\frac{\sigma}{\mu+\alpha} < 1$, the equilibrium (S^o, I^o) is (locally) asymptotically stable.

- Show that if $\frac{B}{\mu}\frac{\sigma}{\mu+\alpha} > 1$, the equilibrium (S^o, I^o) is unstable and the equilibrium (S^*, I^*) is (locally) asymptotically stable.