1. [An SIR model with quarantine]

Consider the following model:

$$\dot{S} = -C(N)\frac{SI}{N}, \quad (1)$$
$$\dot{I} = C(N)\frac{SI}{N} - \gamma I - \alpha I, \quad (2)$$
$$\dot{Q} = \alpha I - \delta Q, \quad (3)$$
$$\dot{R} = \gamma I + \delta Q, \quad (4)$$
$$N = S + I + R. \quad (5)$$

Here, Q is the "density" of infective individuals that are temporary removed from the population (quarantined). $\alpha \geq 0$ is the per-capita rate of removal. We assume that individuals that are quarantined recover at the per-capita rate of $\delta > 0$. We also assume that the quarantined individuals do not take part in the contact process, and hence we set $N = S + I + R$. All other parameters have the same meaning and assumptions as in Section 6 of the lecture notes. All other parameters have the same meaning and assumptions as in Section 6 of the lecture notes. Note also that if $\alpha = 0$, the model reduces to the one in Section 6 of the lecture notes. We define S_0, I_0, Q_0, R_0, and N_0 to be $S(0)$, $I(0)$, $Q(0)$, $R(0)$, and $N(0)$, respectively.

(a) Show that $N(t) + Q(t)$ is a constant, i.e., does not depend on t.

Show that if $Q_0 = 0$, we have $Q(t) = N_0 - N(t)$, and then show that the system above can be rewritten as a system of three differential equations:

$$\dot{N} = -\alpha I + \delta(N_0 - N), \quad (6)$$
$$\dot{S} = -C(N)\frac{SI}{N}, \quad (7)$$
$$\dot{I} = C(N)\frac{SI}{N} - (\gamma + \alpha)I. \quad (8)$$

Solution. The differential equation for N is

$$\dot{N} = \dot{S} + \dot{I} + \dot{R} = -\alpha I + \delta Q,$$

and since $\dot{Q} = \alpha I - \delta Q$, we have a$\dot{N} + \dot{Q} = 0$. Hence $Q(t) + N(t) = Q_0 + N_0$ for all t where the solution is defined. Since $Q_0 = 0$, we have $Q(t) = N_0 - N(t)$ for all t where the solution is defined.

The equations for S and I remain the same, and so we have the system:

$$\dot{N} = -\alpha I + \delta(N_0 - N), \quad (9)$$
$$\dot{S} = -C(N)\frac{SI}{N}, \quad (10)$$
$$\dot{I} = C(N)\frac{SI}{N} - (\gamma + \alpha)I. \quad (11)$$

(b,c) Show that the model is well-defined on the state-space

$$\mathcal{S} = \{(N, S, I) \in \mathbb{R}^3 \ \mid \ S, I > 0, \ S + I \leq N \leq N_0\}. \quad (12)$$

Solution. We define $\mathcal{S}' := \{(N,S,I) \in \mathbb{R}^3 \mid S,I,N > 0\}$ which is an open subset of $\mathbb{R}^3$ and is a superset of $\mathcal{S}$. The vector field

$$f(N,S,I) = \begin{pmatrix} -\alpha I + \delta(N_0 - N) \\ -C(N)\frac{SI}{N} \\ C(N)\frac{SI}{N} - (\gamma+\alpha)I \end{pmatrix}$$

is continuously differentiable on $\mathcal{S}'$, and so there exists a unique solution to the initial value problem (6)–(8) with $(N_0, S_0, I_0) \in \mathcal{S}'$ defined on $t \in [0,b)$ for some $0 < b \leq \infty$.

We now restrict our attention to the case that $(N_0, S_0, I_0) \in \mathcal{S}$.

We have $S(t) = S_0 e^{-\int_0^t C(N(s))\frac{I(s)}{N(s)}\,\mathrm{d}s} > 0$ and $I(t) = I_0 e^{\int_0^t (C(N(s))\frac{I(s)}{N(s)} - \gamma - \alpha)\,\mathrm{d}s} > 0$ for all $t \in [0,b)$. We have $\frac{\mathrm{d}}{\mathrm{d}t}(N_0 - N) = \alpha I - \delta(N_0 - N) > -\delta(N_0 - N)$, and so $N_0 - N(t) \geq (N_0 - N(0))e^{-\delta t} \geq 0$, i.e., $N(t) \leq N_0$ for all $t \in [0,b)$. We have $\dot{N} - \dot{S} - \dot{I} = \delta(N_0 - N) + \gamma I \geq 0$, and so $N(t) - S(t) - I(t) \geq N_0 - S_0 - I_0$, and since $S_0 + I_0 \leq N_0$, we have $S(t) + I(t) \leq N(t)$ for all $t \in [0,b)$. Therefore, the solution with $(N_0, S_0, I_0) \in \mathcal{S}$ stays in $\mathcal{S}$ for $t \in [0,b)$.

Now suppose, to get a contradiction, that $b < \infty$. Then either (a) $|N(t) + S(t) + I(t)| \to \infty$ as $t \to b$, or (b) there exists a sequence (t_k) in $[0,b)$ such that $t_k \to b$ as $k \to \infty$, and such that $\lim_{k\to\infty}(N(t_k), S(t_k), I(t_k))$ exists but is not in $\mathcal{S}'$. (a) is impossible since $N(t) + S(t) + I(t) \leq 2N(t) \leq 2N_0$, and (b) is impossible since $S(t) = S_0 e^{-\int_0^t C(N(s))\frac{I(s)}{N(s)}\,\mathrm{d}s} \geq S_0 e^{-\int_0^t C(N(s))\,\mathrm{d}s} \geq S_0 e^{-\int_0^t C(N_0)\,\mathrm{d}s} \geq S_0 e^{-C(N_0)b}$ and $I(t) = I_0 e^{\int_0^t (C(N(s))\frac{I(s)}{N(s)} - \gamma - \alpha)\,\mathrm{d}s} \geq I_0 e^{-\int_0^t (\gamma+\alpha)\,\mathrm{d}s} \geq I_0 e^{-(\gamma+\alpha)b}$ for all $t \in [0,b)$.

(d,e) Show that $\lim_{t\to\infty} I(t) = 0$ and $\lim_{t\to\infty}\big(N_0 - N(t)\big) = 0$, i.e., $\lim_{t\to\infty} Q(t) = 0$ (and so, $N_\infty := \lim_{t\to\infty} N(t) = N_0$).

Solution. Let $P(t) = S(t) + I(t)$.

Since $\dot{P} = -(\gamma+\alpha)I$, $P(t)$ is positive and decreasing. Hence $P_\infty := \lim_{t\to\infty} P(t)$ exists. Since $|\ddot{P}| = (\gamma+\alpha)|\dot{I}| \leq (\gamma+\alpha)\big|C(N)\frac{SI}{N} - (\gamma+\alpha)I\big| \leq (\gamma+\alpha)\big(C(N)\frac{SI}{N} + (\gamma+\alpha)I\big) \leq (\gamma+\alpha)\big(C(N_0)N_0 + (\gamma+\alpha)N_0\big) < \infty$, $\dot{P}$ is uniformly continuous. From Barbalat's Lemma, we have $\dot{P}(t) \to 0$ as $t \to \infty$. Hence $\lim_{t\to\infty} I(t) = -\lim_{t\to\infty}\dot{P}(t)/(\gamma+\alpha) = 0$.

We now note that $N(t) - P(t) = N(t) - S(t) - I(t)$ is bounded above by N_0. Since $\dot{N} - \dot{P} = \delta(N_0 - N) + \gamma I > 0$, $N(t) - P(t)$ is increasing and so the limit $\lim_{t\to\infty}\big(N(t) - P(t)\big)$ exists. Since $\lim_{t\to\infty} P(t)$ exists, the limit $N_\infty := \lim_{t\to\infty} N(t)$ also exists. Since $|\ddot{N}| \leq \alpha\dot{I} + \delta\dot{N} \leq \alpha\big(C(N)\frac{SI}{N} + (\gamma+\alpha)I\big) + \delta\big(\alpha I + \gamma(N_0 - N)\big) \leq \alpha\big(C(N_0)N_0 + (\gamma+\alpha)N_0\big) + \delta\big(\alpha N_0 + \gamma N_0\big) < \infty$. We can apply Barbalat's Lemma and conclude that $\lim_{t\to\infty}\dot{N} = 0$. Since we know from the last paragraph that $\lim_{t\to\infty} I = 0$, we have $\lim_{t\to\infty}(N_0 - N(t)) = \lim_{t\to\infty}\frac{1}{\delta}\big(\dot{N}(t) + \alpha I(t)\big) = 0$.

(f) Assume that $C(N)/N$ is a decreasing function of N. Show that S_∞ satisfies the final size inequality:

$$\log S_0 - \log S_\infty \geq \frac{C(N_0)}{N_0}\frac{S_0 + I_0 - S_\infty}{\gamma+\alpha}.$$

Solution. Since $\dot{S} + \dot{I} = -(\gamma+\alpha)I$, we have $S(t) + I(t) - S_0 - I_0 = -(\gamma+\alpha)\int_0^t I(s)\,\mathrm{d}s$. Since $\frac{\dot{S}}{S} = -C(N)\frac{I}{N}$, we have

$$\log\frac{S_0}{S(t)} = \int_0^t C(N(s))\frac{I(s)}{N(s)}\,\mathrm{d}s \geq \frac{C(N_0)}{N_0}\int_0^t I(s)\,\mathrm{d}s$$

$$= \frac{C(N_0)}{N_0} \frac{S_0 + I_0 - S(t) - I(t)}{\gamma + \alpha}$$

We know that S_∞ exists because $S(t)$ is a bounded decreasing function. The expression $\frac{C(N_0)}{N_0} \frac{S_0+I_0-S(t)-I(t)}{\gamma+\alpha}$ converges to a finite limit as $t \to \infty$, and this means that $\log \frac{S_0}{S(t)}$ is bounded from above, and hence $S_\infty > 0$. Now we can take the limit of above and get

$$\log \frac{S_0}{S_\infty} \geq \frac{C(N_0)}{N_0} \frac{S_0 + I_0 - S_\infty}{\gamma + \alpha}$$

(g) What is a formula of $\mathcal{R}_0$ for this model?

Solution. From (8), we have $\dot{I} = \big(C(N)\frac{S}{N} - (\gamma + \alpha)\big)I$, Hence the mean infectious period for an infective individual is $\frac{1}{\gamma+\alpha}$. At the beginning, the rate of new infections per unit time per the density of infective individuals is $\frac{C(N_0)S_0}{N_0}$. Hence $\mathcal{R}_0 = \frac{C(N_0)S_0}{N_0(\gamma+\alpha)}$.

The final size inequality can then be written as

$$\frac{S_\infty}{S_0} - \frac{1}{\mathcal{R}_0} \log \frac{S_\infty}{S_0} \geq 1 + \frac{I_0}{S_0}$$

2. Suppose that the probability distribution of the number of secondary infections that a single infective make in its infection period is $\{q_k\}_{k=0}^{\infty}$. Assume that we are interested in the initial phase of the outbreak and so the density of susceptibles and of total population are approximately constant.

(a) Let $q_0 = 1/5$, $q_1 = 1/5$ $q_2 = 2/5$, $q_3 = 1/5$, and $q_k = 0$ for $k \geq 4$. Calculate $\mathcal{R}_0$ and the probability that a single infective introduced into the population will cause an outbreak.

Solution. The probability generating function is

$$g(z) = \sum_{k=0}^{\infty} q_k z^k = \frac{1}{5} + \frac{1}{5}z + \frac{2}{5}z^2 + \frac{1}{5}z^3.$$

The probability z_∞ that an outbreak will *not* occur is the smallest solution in $[0, 1]$ of the equation

$$z_\infty = g(z_\infty) = \frac{1}{5} + \frac{1}{5}z_\infty + \frac{2}{5}z_\infty^2 + \frac{1}{5}z_\infty^3,$$

which can be rewritten as

$$0 = \frac{1}{5} - \frac{4}{5}z_\infty + \frac{2}{5}z_\infty^2 + \frac{1}{5}z_\infty^3 = \frac{1}{5}(z_\infty - 1)(z_\infty^2 + 3z_\infty - 1).$$

The solution of $z^2 + 3z - 1 = 0$ is $-\frac{3}{2} + \frac{\sqrt{13}}{2}, -\frac{3}{2} - \frac{\sqrt{13}}{2} \approx 0.303, -3.303$. Hence $z_\infty = -\frac{3}{2} + \frac{\sqrt{13}}{2} \approx 0.303$. The probability of an outbreak is $1 - z_\infty = \frac{5}{2} - \frac{\sqrt{13}}{2} \approx 0.697$.

(b) Do the same for $q_0 = 2/5$, $q_1 = 2/5$ $q_2 = 1/10$, $q_3 = 1/10$, and $q_k = 0$ for $k \geq 4$.

Solution. The probability generating function is

$$g(z) = \sum_{k=0}^{\infty} q_k z^k = \frac{2}{5} + \frac{2}{5}z + \frac{1}{10}z^2 + \frac{1}{10}z^3.$$

The probability z_∞ that an outbreak will *not* occur is the smallest solution in $[0,1]$ of the equation

$$z_\infty = g(z_\infty) = \frac{2}{5} + \frac{2}{5}z_\infty + \frac{1}{10}z_\infty^2 + \frac{1}{10}z_\infty^3,$$

which can be rewritten as

$$0 = \frac{2}{5} - \frac{3}{5}z_\infty + \frac{1}{10}z_\infty^2 + \frac{1}{10}z_\infty^3 = \frac{1}{10}(z_\infty - 1)(z_\infty^2 + 2z_\infty - 4).$$

The solution of $z^2 + 2z - 4 = 0$ is $-1+\sqrt{5}, -1-\sqrt{5} \approx 1.236, -3.236$, both of which lie outside $[0,1]$. Hence $z_\infty = 1$. The probability of an outbreak is $1 - z_\infty = 0$ (no possibility of an outbreak).

3. Suppose that the probability distribution of the number of secondary infections that a single infective make in its infectious period is $\{q_k\}_{k=0}^\infty$. Assume that the infectious period have a fixed length $T > 0$, and that the contact process can be described by a Poisson process with rate c, i.e.,

$$\text{Prob}\{k \text{ secondary infections caused by a single infective during the time period } \tau\} = \frac{(c\tau)^k}{k!}e^{-c\tau}.$$

Assume that we are interested in the initial phase of the outbreak and so the density of susceptibles and of total population are approximately constant. Find explicit formula for each q_k. Also find explicit formula for $\mathcal{R}_0$. Show that the probability of an outbreak is $1 - z_\infty$, where z_∞ satisfies the equation $z_\infty = e^{\mathcal{R}_0(z_\infty - 1)}$.

Solution. The number of secondary infections that a single infective make in its (fixed length) infection period has a probability distribution $\{q_k\}_{k=0}^\infty$ where

$$q_k = \frac{(cT)^k}{k!}e^{-cT}.$$

The basic reproductive number $\mathcal{R}_0$ is the expected number of secondary infections that a single infective make in its infection period, and so has the value

$$\mathcal{R}_0 = \sum_{k=1}^\infty kq_k = \sum_{k=1}^\infty k\frac{(cT)^k}{k!}e^{-cT} = cTe^{-cT}\sum_{k=1}^\infty \frac{(cT)^{k-1}}{(k-1)!} = cTe^{-cT}\sum_{k=0}^\infty \frac{(cT)^k}{k!} = cTe^{-cT}e^{cT} = cT.$$

The probability of an outbreak is $1 - z_\infty$, where z_∞ satisfies the equation

$$z_\infty = \sum_{k=0}^\infty q_k z_\infty^k = \sum_{k=0}^\infty \frac{(cTz_\infty)^k}{k!}e^{-cT} = e^{-cT}\sum_{k=0}^\infty \frac{(cTz_\infty)^k}{k!} = e^{-cT}e^{cTz_\infty} = e^{cT(z_\infty - 1)} = e^{\mathcal{R}_0(z_\infty - 1)}.$$