1. [An SIR model with quarantine]

Consider the following model:

$$\dot{S} = -C(N)\frac{SI}{N}, \tag{1}$$
$$\dot{I} = C(N)\frac{SI}{N} - \gamma I - \alpha I, \tag{2}$$
$$\dot{Q} = \alpha I - \delta Q, \tag{3}$$
$$\dot{R} = \gamma I + \delta Q, \tag{4}$$
$$N = S + I + R. \tag{5}$$

Here, Q is the "density" of infective individuals that are temporary removed from the population (quarantined). $\alpha \geq 0$ is the per-capita rate of removal. We assume that individuals that are quarantined recover at the per-capita rate of $\delta > 0$. We also assume that the quarantined individuals do not take part in the contact process, and hence we set $N = S + I + R$. All other parameters have the same meaning and assumptions as in Section 6 of the lecture notes. Note also that if $\alpha = 0$, the model reduces to the one in Section 6 of the lecture notes. We define S_0, I_0, Q_0, R_0, and N_0 to be $S(0)$, $I(0)$, $Q(0)$, $R(0)$, and $N(0)$, respectively.

(a) Show that $N(t) + Q(t)$ is a constant, i.e., does not depend on t.

Show that if $Q_0 = 0$, we have $Q(t) = N_0 - N(t)$, and then show that the system above can be rewritten as a system of three differential equations:

$$\dot{N} = -\alpha I + \delta(N_0 - N), \tag{6}$$
$$\dot{S} = -C(N)\frac{SI}{N}, \tag{7}$$
$$\dot{I} = C(N)\frac{SI}{N} - (\gamma + \alpha)I. \tag{8}$$

(b,c) Show that the model is well-defined on the state-space

$$\mathcal{S} = \{(N, S, I) \in \mathbb{R}^3 \,:\, S, I > 0,\ S + I \leq N \leq N_0\}. \tag{9}$$

Hint: show that for any $(N_0, S_0, I_0) \in \{(N, S, I) \in \mathbb{R}^3 \,:\, S, I, N > 0\}$, there is a unique solution to (6)–(8). Then shows that if $(N_0, S_0, I_0) \in \mathcal{S}$, then the solution stays in $\mathcal{S}$ for all t that the solution is defined. Finally show that the solution is defined for all $t \in [0, \infty)$.

(d,e) Show that $\lim_{t\to\infty} I(t) = 0$ and $\lim_{t\to\infty} \big(N_0 - N(t)\big) = 0$, i.e., $\lim_{t\to\infty} Q(t) = 0$ (and so, $N_\infty := \lim_{t\to\infty} N(t) = N_0$).

Hint: Work with $P(t) = S(t) + I(t)$. Show that $\lim_{t\to\infty} P(t)$ exists and that $\lim_{t\to\infty} \dot{P} = 0$. Then show that $\lim_{t\to\infty}(N(t) - P(t))$ exists, and so $\lim_{t\to\infty} N(t)$ exists. Then show that $\lim_{t\to\infty} \dot{N} = 0$.

(f) Assume that $C(N)/N$ is a decreasing function of N. Show that S_∞ satisfies the final size inequality:

$$\log S_0 - \log S_\infty \geq \frac{C(N_0)}{N_0}\frac{S_0 + I_0 - S_\infty}{\gamma + \alpha}.$$

(g) What is a formula of $\mathcal{R}_0$ for this model?

2. Suppose that the probability distribution of the number of secondary infections that a single infective make in its infection period is $\{q_k\}_{k=0}^{\infty}$. Assume that we are interested in the initial phase of the outbreak and so the density of susceptibles and of total population are approximately constant.

(a) Let $q_0 = 1/5$, $q_1 = 1/5$, $q_2 = 2/5$, $q_3 = 1/5$, and $q_k = 0$ for $k \geq 4$. Calculate $\mathcal{R}_0$ and the probability that a single infective introduced into the population will cause outbreak.

(b) Do the same for $q_0 = 2/5$, $q_1 = 2/5$, $q_2 = 1/10$, $q_3 = 1/10$, and $q_k = 0$ for $k \geq 4$.

Hint: Recall that the probability of an outbreak is $1 - z_\infty$ where z_∞ is the smallest solution in $[0, 1]$ of $z_\infty = g(z_\infty)$, and where $g(z)$ is the probability generating function of $\{q_k\}_{k=0}^{\infty}$.

3. Suppose that the probability distribution of the number of secondary infections that a single infective make in its infectious period is $\{q_k\}_{k=0}^{\infty}$. Assume that the infectious period have a fixed length $T > 0$, and that the contact process can be described by a Poisson process with rate c, i.e.,

$$\text{Prob}\{k \text{ secondary infections caused by a single infective during the time period } \tau\} = \frac{(c\tau)^k}{k!}e^{-c\tau}.$$

Assume that we are interested in the initial phase of the outbreak and so the density of susceptibles and of total population are approximately constant. Find explicit formula for each q_k. Also find explicit formula for $\mathcal{R}_0$. Show that the probability of an outbreak is $1 - z_\infty$, where z_∞ satisfies the equation $z_\infty = e^{\mathcal{R}_0(z_\infty - 1)}$.