Consider an age-of-infection epidemic model with mass-action incidence:

$$\dot{S}(t) = -\phi(t)S(t),\ t \geq 0, \quad S(0) = S_0 \tag{1}$$

$$i(t,\tau) = \begin{cases} -\dot{S}(t-\tau)\mathcal{F}(\tau) & ,0 \leq \tau < t, \\ i_0(\tau - t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)} & ,0 \leq t \leq \tau, \end{cases} \tag{2}$$

$$\phi(t) = \int_0^\infty \sigma(\tau)i(t,\tau)\,\mathrm{d}\tau,\ t \geq 0, \tag{3}$$

where S_0 and i_0 are the initial conditions, and σ and $\mathcal{F}$ are the model's parameters. We make the following the assumptions:

(i) $S_0 > 0$,

(ii) $i_0(\tau) \geq 0, \forall \tau \in [0,\infty)$, $\quad 0 < \int_0^\infty i_0(\tau)\,\mathrm{d}\tau < \infty$,

(iii) $\sigma : [0,\infty) \to [0,\infty)$ is bounded,

(iv) $\mathcal{F} : [0,\infty) \to [0,1]$ is non-increasing, $\mathcal{F}(0) = 1$, and $\int_0^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau < \infty$,

(v) $\int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0$,

(vi) if $\mathcal{F}(c) = 0$ for some $c > 0$, and so $\mathcal{F}(\tau) = 0$ for $\tau \geq c$, then $i_0(\tau) = 0$ for $\tau \geq c$, and

(vii) we define $\frac{i_0(\tau)}{\mathcal{F}(\tau)} = 0$ if $i_0(\tau) = \mathcal{F}(\tau) = 0$, and also $\frac{i(t,\tau)}{\mathcal{F}(\tau)} = 0$ if $i(t,\tau) = \mathcal{F}(\tau) = 0$.

Note that:

- τ is the infection age, i.e., the amount of time since an individual first entered the infective class.
- $\int_a^b i(t,\tau)\,\mathrm{d}\tau$ is the density of infectives, at time t, that have infection age between a and b.
- $\mathcal{F}(\tau)$ is the probability that an individual with infection age τ is still in the infective class.
- $\int_0^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau$ is the expected amount time that an infective individual spends in the infective class, and we require that this amount is finite.
- $\sigma(\tau)$ is the "infection rate" of an individual with infection age τ. More precisely, $S(t)\int_a^b \sigma(\tau)i(t,\tau)\,\mathrm{d}\tau$ is the rate of new infections, at time t, caused by infectives that have infection age between a and b.
- $\phi(t)$ is the so-called *force of infection* at time t: the per-capita rate at time t that susceptible individuals contract the diseases). Since $\phi(t) = -\int_0^t \sigma(\tau)\dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_t^\infty \sigma(\tau)i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau$, we require that $\int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0$. Otherwise we have $\phi(t) = 0$ for all $t \geq 0$, and the disease cannot spread at all.
- S_0 is the initial density of susceptibles, and $\int_a^b i_0(\tau)\,\mathrm{d}\tau$ is the initial density of infectives with infection age between a and b.

A (non-negative) solution of (1)–(3) is given by two functions, $S : [0,\infty) \to [0,\infty)$ and $i : [0,\infty) \times [0,\infty) \to [0,\infty)$, such that the function ϕ is continuous (and so S is continuously differentiable), $\int_0^\infty i(t,\tau)\,\mathrm{d}\tau < \infty$ for each $t \geq 0$, and such that (1)–(3) are satisfied.

In this homework, you can assume that there always exists a unique solution of (1)–(3).

1. Suppose that $\mathcal{F}(c) = 0$ for some $c > 0$ (and so $\mathcal{F}(\tau) = 0$ for all $\tau \geq c$). Show that $i(t,\tau) = 0$ for all $t \geq 0$ and all $\tau \geq c$.

We fix $\tau \geq c$. Hence $\mathcal{F}(\tau) = 0$. From (2), if $\tau < t$, then $i(t,\tau) = -\dot{S}(t-\tau)\mathcal{F}(\tau) = 0$, and if $0 \leq t \leq \tau$, then $i(t,\tau) = i_0(\tau - t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)} = 0$. Recall that $\frac{i_0(\tau-t)}{\mathcal{F}(\tau-t)}$ is defined even when $\mathcal{F}(\tau - t) = 0$ by the assumptions (vi) and (vii).

2. Recall the equation for the final size S_∞ of the epidemic model (1)–(3):

$$S_\infty - \frac{1}{\kappa}\log S_\infty = S_0 - \frac{1}{\kappa}\log S_0 + y_0 \tag{4}$$

where

$$\kappa = \int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0, \text{ and} \tag{5}$$

$$y_0 = \frac{1}{\kappa}\int_0^\infty \frac{i_0(\tau)}{\mathcal{F}(\tau)}\int_\tau^\infty \sigma(r)\mathcal{F}(r)\,\mathrm{d}r\,\mathrm{d}\tau. \tag{6}$$

Here, $S_\infty := \lim_{t\to\infty} S(t)$. We define

$$\mathcal{R}_0 = S_0\kappa. \tag{7}$$

Show that

(a) If $\mathcal{R}_0 \leq 1$, then $S_\infty \to S_0$ as $y_0 \to 0$.

Since $S_0\kappa = \mathcal{R}_0 \leq 1$, we have $S_0 \leq \frac{1}{\kappa}$. Define the function $f(x) = x - \frac{1}{\kappa}\log x$ on $x \in (0, \frac{1}{\kappa}]$. Then f is strictly decreasing and has continuous inverse. The equation (4) can be written as

$$f(S_\infty) = f(S_0) + y_0,$$

and so

$$S_\infty = f^{-1}(f(S_\infty)) = f^{-1}(f(S_0) + y_0).$$

Hence

$$\lim_{y_0\to 0} S_\infty = \lim_{y_0\to 0} f^{-1}(f(S_0) + y_0) = f^{-1}(f(S_0)) = S_0.$$

(b) If $\mathcal{R}_0 > 1$, then $S_0 - S_\infty$ stays bounded away from 0 as $y_0 \to 0$.

Since $S_0\kappa = \mathcal{R}_0 > 1$, we have $S_0 > \frac{1}{\kappa}$. Define the function $f(x) = x - \frac{1}{\kappa}\log x$ on $x \in (0,\infty)$. Then f is concave up and has a unique minimum at $x = \frac{1}{\kappa}$. The equation (4) can be written as

$$f(S_\infty) = f(S_0) + y_0,$$

and S_∞ is a unique real number in $(0, \frac{1}{\kappa})$ that satisfies this equation (cf. Homework #2, Problem 3). Let $x \in (0, \frac{1}{\kappa})$ be such that $f(x) = f(S_0)$. We then have $S_\infty < x < \frac{1}{\kappa} < S_0$ (since f is strictly decreasing on $(0, \frac{1}{\kappa}]$). Let $a = S_0 - x > 0$. Note that the value of a does not depend on the value of y_0. Hence $S_0 - S_\infty = (S_0 - x) + (x - S_\infty) > a$ for all $y_0 > 0$.

3. Assume that $\mathcal{F}$ is continuously differentiable and denote its derivative by $\mathcal{F}'$. Let $R : [0, \infty) \to [0, \infty)$ be a solution of the initial value problem:

$$\dot{R}(t) = -\int_0^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau, \quad R(0) = 0. \tag{8}$$

The function $R(t)$ gives the density of individuals at time t that are recovered or removed. Assume that the function $t \mapsto \int_0^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau$ is continuous, and so the function R is well-defined and is continuously differentiable (in fact, $R(t) = -\int_0^t\int_0^\infty i(s,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s$).

Show that $N := S(t) + \int_0^\infty i(t,\tau)\,\mathrm{d}\tau + R(t)$ is constant. Feel free to take derivatives under integral signs, if needed.

We have

$$\begin{aligned}\int_0^\infty i(t,\tau)\,\mathrm{d}\tau &= -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_t^\infty i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau \\ &= -\int_0^t \dot{S}(\tau)\mathcal{F}(t-\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(t+\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau,\end{aligned}$$

and so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,\tau)\,\mathrm{d}\tau &= -\dot{S}(t)\mathcal{F}(0) - \int_0^t \dot{S}(\tau)\mathcal{F}'(t-\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}'(t+\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau \\ &= -\dot{S}(t) - \int_0^t \dot{S}(t-\tau)\mathcal{F}'(\tau)\,\mathrm{d}\tau + \int_t^\infty i_0(\tau-t)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau.\end{aligned}$$

From (2), we have $-\dot{S}(t-\tau) = \frac{i(t,\tau)}{\mathcal{F}(\tau)}$ for $0 \le \tau < t$ and $\frac{i_0(\tau-t)}{\mathcal{F}(\tau-t)} = \frac{i(t,\tau)}{\mathcal{F}(\tau)}$ for $0 \le t \le \tau$ Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,\tau)\,\mathrm{d}\tau &= -\dot{S}(t) + \int_0^t i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau + \int_t^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau \\ &= -\dot{S}(t) + \int_0^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau \\ &= -\dot{S}(t) - \dot{R}(t).\end{aligned}$$

4. [An epidemic model with fixed latent period] Let $c > 0$ and $\bar{\sigma} > 0$ be fixed positive real numbers. Let

$$\mathcal{F}(\tau) = \begin{cases} 1 & , 0 \le \tau \le c, \\ e^{-\gamma(\tau-c)} & , \tau > c, \end{cases} \tag{9}$$

and

$$\sigma(\tau) = \begin{cases} 0 & , 0 \le \tau \le c, \\ \bar{\sigma} & , \tau > c. \end{cases} \tag{10}$$

We interpret c as the length of latent (exposed) period where an individual is infected but is still not capable of spreading the diseases. After the latent period, the infective individual spreads the disease with the (mass-action) rate $\bar{\sigma}$, and the probability that the individual still stays infective is exponentially distributed with the parameter $-\gamma$.

(a) Calculate the expected amount of time that an infective individual spends in the infective class, and also the expected amount of time that an infective individual is capable of spreading the disease.

The expected amount time that an infective individual spends in the infective class is given by

$$\int_0^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau = \int_0^c \mathcal{F}(\tau)\,\mathrm{d}\tau + \int_c^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau = c + \int_c^\infty e^{-\gamma(\tau-c)}\,\mathrm{d}\tau = c + \int_0^\infty e^{-\gamma\tau}\,\mathrm{d}\tau = c + \frac{1}{\gamma}.$$

The expected amount time that an infective individual spends spreading the disease is given by

$$\int_c^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau = \int_c^\infty e^{-\gamma(\tau-c)}\,\mathrm{d}\tau = \int_0^\infty e^{-\gamma\tau}\,\mathrm{d}\tau = \frac{1}{\gamma}.$$

(b) Calculate the explicit expressions for κ and y_0 in (4), simplify the expressions as much as you can. What is this model's $\mathcal{R}_0$?

We have

$$\kappa = \int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau = \bar{\sigma}\int_c^\infty e^{-\gamma(\tau-c)}\,\mathrm{d}\tau = \bar{\sigma}\int_0^\infty e^{-\gamma\tau}\,\mathrm{d}\tau = \frac{\bar{\sigma}}{\gamma},$$

and

$$\begin{aligned}
y_0 &= \frac{1}{\kappa}\int_0^\infty \frac{i_0(\tau)}{\mathcal{F}(\tau)}\int_\tau^\infty \sigma(r)\mathcal{F}(r)\,\mathrm{d}r\,\mathrm{d}\tau \\
&= \frac{1}{\kappa}\int_0^c \frac{i_0(\tau)}{\mathcal{F}(\tau)}\int_\tau^\infty \sigma(r)\mathcal{F}(r)\,\mathrm{d}r\,\mathrm{d}\tau + \frac{1}{\kappa}\int_c^\infty \frac{i_0(\tau)}{\mathcal{F}(\tau)}\int_\tau^\infty \sigma(r)\mathcal{F}(r)\,\mathrm{d}r\,\mathrm{d}\tau \\
&= \frac{\bar{\sigma}}{\kappa}\int_0^c \frac{i_0(\tau)}{e^{-\gamma(\tau-c)}}\int_c^\infty e^{-\gamma(r-c)}\,\mathrm{d}r\,\mathrm{d}\tau + \frac{\bar{\sigma}}{\kappa}\int_c^\infty \frac{i_0(\tau)}{e^{-\gamma(\tau-c)}}\int_\tau^\infty e^{-\gamma(r-c)}\,\mathrm{d}r\,\mathrm{d}\tau \\
&= \frac{\bar{\sigma}}{\kappa}\int_0^c \frac{i_0(\tau)}{e^{-\gamma(\tau-c)}}\frac{1}{\gamma}\,\mathrm{d}\tau + \frac{\bar{\sigma}}{\kappa}\int_c^\infty \frac{i_0(\tau)}{e^{-\gamma(\tau-c)}}\frac{e^{-\gamma(\tau-c)}}{\gamma}\,\mathrm{d}\tau \\
&= \frac{\bar{\sigma}}{\kappa\gamma}\int_0^c i_0(\tau)e^{\gamma(\tau-c)}\,\mathrm{d}\tau + \frac{\bar{\sigma}}{\kappa\gamma}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau \\
&= \int_0^c i_0(\tau)e^{\gamma(\tau-c)}\,\mathrm{d}\tau + \int_c^\infty i_0(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

The model's $\mathcal{R}_0$ is $S_0\kappa = \frac{\bar{\sigma}S_0}{\gamma}$. Note that this is exactly the same $\mathcal{R}_0$ as in the simplified Kermack-McKendrick model. Does this seem reasonable to you?

(c) Define a function $E : [0,\infty) \to [0,\infty)$ by

$$E(t) = \int_0^c i(t,\tau)\,\mathrm{d}\tau, \quad t \geq 0. \tag{11}$$

The function $E(t)$ gives the density of individuals at time t that are infected but are still not capable of spreading the diseases (the "exposed" class).

Define a function $I : [0,\infty) \to [0,\infty)$ by

$$I(t) = \int_c^\infty i(t,\tau)\,\mathrm{d}\tau, \quad t \geq 0, \tag{12}$$

(13)

The function $I(t)$, for $t \geq 0$, gives the density of individuals at time t that are infected and are capable of spreading the diseases.

Define $R : [0, \infty) \to [0, \infty)$ to be the solution of the initial value problem:

$$\dot{R}(t) = -\int_c^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau, \quad R(0) = 0. \tag{14}$$

The function $R(t)$ gives the density of individuals at time t that are recovered or removed.

Assume that the function E and I are continuous. Assume that the function $t \mapsto \int_c^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau$ is continuous, and so the function R is well-defined and is continuously differentiable (in fact, $R(t) = -\int_0^t \int_c^\infty i(s,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s$).

Show that, for $t \geq 0$, the solution of (1)–(3) satisfies the system of delay differential equations:

$$\dot{S}(t) = -\bar{\sigma}S(t)I(t), \tag{15}$$

$$\dot{E}(t) = \begin{cases} \bar{\sigma}S(t)I(t) - \bar{\sigma}S(t-c)I(t-c) & ,t > c, \\ \bar{\sigma}S(t)I(t) - i_0(c-t) & ,t < c, \end{cases} \tag{16}$$

$$\dot{I}(t) = \begin{cases} \bar{\sigma}S(t-c)I(t-c) - \gamma I(t) & ,t > c \\ -\gamma I(t) + i_0(c-t) & ,t < c, \end{cases} \tag{17}$$

$$\dot{R}(t) = \gamma I(t). \tag{18}$$

We have

$$\phi(t) = \int_0^\infty \sigma(\tau)i(t,\tau)\,\mathrm{d}\tau = \bar{\sigma}\int_c^\infty i(t,\tau)\,\mathrm{d}\tau = \bar{\sigma}I(t), \quad t \geq 0,$$

and so

$$\dot{S}(t) = -\phi(t)S(t) = -\bar{\sigma}S(t)I(t), \quad t \geq 0.$$

We have

$$\dot{R}(t) = -\int_c^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau = \gamma\int_c^\infty i(t,\tau)\frac{e^{-\gamma(\tau-c)}}{e^{-\gamma(\tau-c)}}\,\mathrm{d}\tau = \gamma I(t), \quad t \geq 0.$$

We now calculate $I(t)$, and then $\dot{I}(t)$:

$$\begin{aligned} I(t) = \int_c^\infty i(t,\tau)\,\mathrm{d}\tau &= \begin{cases} \int_c^t i(t,\tau)\,\mathrm{d}\tau + \int_t^\infty i(t,\tau)\,\mathrm{d}\tau & ,t \geq c \\ \int_c^\infty i(t,\tau)\,\mathrm{d}\tau & ,t < c \end{cases} \\ &= \begin{cases} -\int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_t^\infty i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau & ,t \geq c \\ \int_c^\infty i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau & ,t < c \end{cases} \\ &= \begin{cases} -\int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t \geq c \\ \int_{c-t}^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t < c. \end{cases} \\ &= \begin{cases} -\int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t \geq c \\ \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau - \int_0^{c-t} i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t < c, \end{cases} \end{aligned}$$

$$= \begin{cases} -\int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t \ge c \\ \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau & ,t < c. \end{cases}$$

Let $x(t) = -\int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau$, for $t \ge c$, and let $y(t) = \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau$, for $t \ge 0$. Then

$$I(t) = \begin{cases} x(t) + y(t) & ,t \ge c \\ y(t) - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau & ,t < c. \end{cases}$$

We calculate $x(t)$ and $\dot{x}(t)$:

$$x(t) = -\int_c^t \dot{S}(t-\tau)e^{-\gamma(\tau-c)}\,\mathrm{d}\tau = -\int_0^{t-c} \dot{S}(\tau)e^{-\gamma(t-\tau-c)}\,\mathrm{d}\tau = -e^{-\gamma(t-c)}\int_0^{t-c} \dot{S}(\tau)e^{\gamma\tau}\,\mathrm{d}\tau,$$

and so

$$\dot{x}(t) = \gamma e^{-\gamma(t-c)}\int_0^{t-c} \dot{S}(\tau)e^{\gamma\tau}\,\mathrm{d}\tau - e^{-\gamma(t-c)}\dot{S}(t-c)e^{\gamma(t-c)} = -\gamma x(t) + \bar{\sigma}S(t-c)I(t-c), \quad t > c.$$

And we calculate $y(t)$ and $\dot{y}(t)$:

$$\begin{aligned} y(t) &= \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau \\ &= \begin{cases} \int_0^c i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau + \int_c^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t \ge c \\ \int_0^{c-t} i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau + \int_{c-t}^c i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau + \int_c^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau & ,t < c \end{cases} \\ &= \begin{cases} \int_0^c i_0(\tau)e^{-\gamma(\tau+t-c)}\,\mathrm{d}\tau + \int_c^\infty i_0(\tau)\frac{e^{-\gamma(\tau+t-c)}}{e^{-\gamma(\tau-c)}}\,\mathrm{d}\tau & ,t \ge c \\ \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau + \int_{c-t}^c i_0(\tau)e^{-\gamma(\tau+t-c)}\,\mathrm{d}\tau + \int_c^\infty i_0(\tau)\frac{e^{-\gamma(\tau+t-c)}}{e^{-\gamma(\tau-c)}}\,\mathrm{d}\tau & ,t < c \end{cases} \\ &= \begin{cases} e^{-\gamma(t-c)}\int_0^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau + e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau & ,t \ge c \\ \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau + e^{-\gamma(t-c)}\int_{c-t}^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau + e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau & ,t < c, \end{cases} \end{aligned}$$

and so

$$\begin{aligned} \dot{y}(t) &= \begin{cases} -\gamma e^{-\gamma(t-c)}\int_0^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau - \gamma e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau & ,t > c \\ -i_0(c-t) - \gamma e^{-\gamma(t-c)}\int_{c-t}^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau + e^{-\gamma(t-c)}i_0(c-t)e^{-\gamma(c-t)} - \gamma e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau & ,t < c \end{cases} \\ &= \begin{cases} -\gamma\big(e^{-\gamma(t-c)}\int_0^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau + e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau\big) & ,t > c \\ -\gamma\big(e^{-\gamma(t-c)}\int_{c-t}^c i_0(\tau)e^{-\gamma\tau}\,\mathrm{d}\tau + e^{-\gamma t}\int_c^\infty i_0(\tau)\,\mathrm{d}\tau\big) & ,t < c \end{cases} \\ &= \begin{cases} -\gamma y(t) & ,t > c \\ -\gamma\big(y(t) - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau\big) & ,t < c. \end{cases} \end{aligned}$$

Since

$$I(t) = \begin{cases} x(t) + y(t) & ,t \ge c \\ y(t) - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau & ,t < c, \end{cases}$$

we have

$$\dot{I}(t) = \begin{cases} \dot{x}(t) + \dot{y}(t) & ,t > c \\ \dot{y}(t) + i_0(c-t) & ,t < c \end{cases}$$

$$= \begin{cases} \bar{\sigma}S(t-c)I(t-c) - \gamma(x(t)+y(t)) & ,t>c \\ -\gamma\big(y(t) - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau\big) + i_0(c-t) & ,t<c \end{cases}$$

$$= \begin{cases} \bar{\sigma}S(t-c)I(t-c) - \gamma I(t) & ,t>c \\ -\gamma I(t) + i_0(c-t) & ,t<c. \end{cases}$$

We now calculate $E(t)+I(t)$, and then $\dot{E}(t)+\dot{I}(t)$:

$$\begin{aligned} E(t)+I(t) &= \int_0^\infty i(t,\tau)\,\mathrm{d}\tau = \int_0^t i(t,\tau)\,\mathrm{d}\tau + \int_t^\infty i(t,\tau)\,\mathrm{d}\tau \\ &= -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_t^\infty i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau \\ &= -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(\tau+t)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau. \end{aligned}$$

Let $z(t) = -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau$. Then

$$E(t)+I(t) = z(t)+y(t).$$

We calculate $z(t)$ and $\dot{z}(t)$:

$$\begin{aligned} z(t) &= -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau \\ &= \begin{cases} -\int_0^c \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau - \int_c^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau & ,t\geq c \\ -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau & ,t<c \end{cases} \\ &= \begin{cases} -\int_0^c \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + x(t) & ,t\geq c \\ -\int_0^t \dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau & ,t<c \end{cases} \\ &= \begin{cases} -\int_0^c \dot{S}(t-\tau)\,\mathrm{d}\tau + x(t) & ,t\geq c \\ -\int_0^t \dot{S}(t-\tau)\,\mathrm{d}\tau & ,t<c \end{cases} \\ &= \begin{cases} -\int_{t-c}^t \dot{S}(\tau)\,\mathrm{d}\tau + x(t) & ,t\geq c \\ -\int_0^t \dot{S}(\tau)\,\mathrm{d}\tau & ,t<c, \end{cases} \end{aligned}$$

and so

$$\begin{aligned} \dot{z}(t) &= \begin{cases} -\dot{S}(t) + \dot{S}(t-c) + \dot{x}(t) & ,t>c \\ -\dot{S}(t) & ,t<c \end{cases} \\ &= \begin{cases} \bar{\sigma}S(t)I(t) - \bar{\sigma}S(t-c)I(t-c) - \gamma x(t) + \bar{\sigma}S(t-c)I(t-c) & ,t>c \\ \bar{\sigma}S(t)I(t) & ,t<c \end{cases} \\ &= \begin{cases} \bar{\sigma}S(t)I(t) - \gamma x(t) & ,t>c \\ \bar{\sigma}S(t)I(t) & ,t<c. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \dot{E}(t)+\dot{I}(t) &= \dot{z}(t)+\dot{y}(t) \\ &= \begin{cases} \bar{\sigma}S(t)I(t) - \gamma\big((x(t)+y(t)\big) & ,t>c \\ \bar{\sigma}S(t)I(t) - \gamma\big(y(t) - \int_0^{c-t} i_0(\tau)\,\mathrm{d}\tau\big) & ,t<c \end{cases} \end{aligned}$$

$$= \begin{cases} \bar{\sigma}S(t)I(t) - \gamma I(t) & ,t > c \\ \bar{\sigma}S(t)I(t) - \gamma I(t) & ,t < c, \end{cases}$$
$$= \bar{\sigma}S(t)I(t) - \gamma I(t) \quad t \neq c.$$

Finally, we have

$$\dot{E}(t) = \big(\dot{E}(t) + \dot{I}(t)\big) - \dot{I}(t) = \begin{cases} \bar{\sigma}S(t)I(t) - \bar{\sigma}S(t-c)I(t-c) & ,t > c \\ \bar{\sigma}S(t)I(t) - i_0(c-t) & ,t < c. \end{cases}$$