Consider an age-of-infection epidemic model with mass-action incidence:

$$\dot{S}(t) = -\phi(t)S(t),\ t \geq 0, \quad S(0) = S_0 \tag{1}$$

$$i(t,\tau) = \begin{cases} -\dot{S}(t-\tau)\mathcal{F}(\tau) & ,0 \leq \tau < t, \\ i_0(\tau - t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)} & ,0 \leq t \leq \tau, \end{cases} \tag{2}$$

$$\phi(t) = \int_0^\infty \sigma(\tau)i(t,\tau)\,\mathrm{d}\tau,\ t \geq 0, \tag{3}$$

where S_0 and i_0 are the initial conditions, and σ and $\mathcal{F}$ are the model's parameters. We make the following the assumptions:

(i) $S_0 > 0$,

(ii) $i_0(\tau) \geq 0, \forall \tau \in [0,\infty), \quad 0 < \int_0^\infty i_0(\tau)\,\mathrm{d}\tau < \infty$,

(iii) $\sigma : [0,\infty) \to [0,\infty)$ is bounded,

(iv) $\mathcal{F} : [0,\infty) \to [0,1]$ is non-increasing, $\mathcal{F}(0) = 1$, and $\int_0^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau < \infty$,

(v) $\int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0$,

(vi) if $\mathcal{F}(c) = 0$ for some $c > 0$, and so $\mathcal{F}(\tau) = 0$ for $\tau \geq c$, then $i_0(\tau) = 0$ for $\tau \geq c$, and

(vii) we define $\frac{i_0(\tau)}{\mathcal{F}(\tau)} = 0$ if $i_0(\tau) = \mathcal{F}(\tau) = 0$, and also $\frac{i(t,\tau)}{\mathcal{F}(\tau)} = 0$ if $i(t,\tau) = \mathcal{F}(\tau) = 0$.

Note that:

- τ is the infection age, i.e., the amount of time since an individual first entered the infective class.
- $\int_a^b i(t,\tau)\,\mathrm{d}\tau$ is the density of infectives, at time t, that have infection age between a and b.
- $\mathcal{F}(\tau)$ is the probability that an individual with infection age τ is still in the infective class.
- $\int_0^\infty \mathcal{F}(\tau)\,\mathrm{d}\tau$ is the expected amount time that an infective individual spends in the infective class, and we require that this amount is finite.
- $\sigma(\tau)$ is the "infection rate" of an individual with infection age τ. More precisely, $S(t)\int_a^b \sigma(\tau)i(t,\tau)\,\mathrm{d}\tau$ is the rate of new infections, at time t, caused by infectives that have infection age between a and b.
- $\phi(t)$ is the so-called *force of infection* at time t: the per-capita rate at time t that susceptible individuals contract the diseases). Since $\phi(t) = -\int_0^t \sigma(\tau)\dot{S}(t-\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau + \int_t^\infty \sigma(\tau)i_0(\tau-t)\frac{\mathcal{F}(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau$, we require that $\int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0$. Otherwise we have $\phi(t) = 0$ for all $t \geq 0$, and the disease cannot spread at all.
- S_0 is the initial density of susceptibles, and $\int_a^b i_0(\tau)\,\mathrm{d}\tau$ is the initial density of infectives with infection age between a and b.

A (non-negative) solution of (1)–(3) is given by two functions, $S : [0,\infty) \to [0,\infty)$ and $i : [0,\infty) \times [0,\infty) \to [0,\infty)$, such that the function ϕ is continuous (and so S is continuously differentiable), $\int_0^\infty i(t,\tau)\,\mathrm{d}\tau < \infty$ for each $t \geq 0$, and such that (1)–(3) are satisfied.

In this homework, you can assume that there always exists a unique solution of (1)–(3).

1. Suppose that $\mathcal{F}(c) = 0$ for some $c > 0$ (and so $\mathcal{F}(\tau) = 0$ for all $\tau \geq c$). Show that $i(t,\tau) = 0$ for all $t \geq 0$ and all $\tau \geq c$.

Hint: Use (2) and the assumptions (vi) and (vii).

2. Recall the equation for the final size S_∞ of the epidemic model (1)–(3):

$$S_\infty - \frac{1}{\kappa}\log S_\infty = S_0 - \frac{1}{\kappa}\log S_0 + y_0 \tag{4}$$

where

$$\kappa = \int_0^\infty \sigma(\tau)\mathcal{F}(\tau)\,\mathrm{d}\tau > 0, \text{ and} \tag{5}$$

$$y_0 = \frac{1}{\kappa}\int_0^\infty \frac{i_0(\tau)}{\mathcal{F}(\tau)}\int_\tau^\infty \sigma(r)\mathcal{F}(r)\,\mathrm{d}r\,\mathrm{d}\tau. \tag{6}$$

Here, $S_\infty := \lim_{t\to\infty} S(t)$. We define

$$\mathcal{R}_0 = S_0\kappa. \tag{7}$$

Show that

(a) If $\mathcal{R}_0 \leq 1$, then $S_\infty \to S_0$ as $y_0 \to 0$.

(b) If $\mathcal{R}_0 > 1$, then $S_0 - S_\infty$ stays bounded away from 0 as $y_0 \to 0$.

3. Assume that $\mathcal{F}$ is continuously differentiable and denote its derivative by $\mathcal{F}'$. Let $R : [0,\infty) \to [0,\infty)$ be a solution of the initial value problem:

$$\dot{R}(t) = -\int_0^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau, \quad R(0) = 0. \tag{8}$$

The function $R(t)$ gives the density of individuals at time t that are recovered or removed. Assume that the function $t \mapsto \int_0^\infty i(t,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau$ is continuous, and so the function R is well-defined and is continuously differentiable (in fact, $R(t) = -\int_0^t\int_0^\infty i(s,\tau)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s$).

Show that $N := S(t) + \int_0^\infty i(t,\tau)\,\mathrm{d}\tau + R(t)$ is constant. Feel free to take derivatives under integral signs, if needed.

Hint:

(a) Show that $\int_0^\infty i(t,\tau)\,\mathrm{d}\tau = -\int_0^t \dot{S}(\tau)\mathcal{F}(t-\tau)\,\mathrm{d}\tau + \int_0^\infty i_0(\tau)\frac{\mathcal{F}(t+\tau)}{\mathcal{F}(\tau)}\,\mathrm{d}\tau$.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,\tau)\,\mathrm{d}\tau = -\dot{S}(t) - \int_0^t \dot{S}(t-\tau)\mathcal{F}'(\tau)\,\mathrm{d}\tau + \int_t^\infty i_0(\tau - t)\frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau-t)}\,\mathrm{d}\tau$.

(c) Use (2) show that $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,\tau)\,\mathrm{d}\tau = -\dot{S}(t) - \dot{R}(t)$.

4. [An epidemic model with fixed latent period] Let $c > 0$ and $\bar{\sigma} > 0$ be fixed positive real numbers. Let

$$\mathcal{F}(\tau) = \begin{cases} 1 & , 0 \leq \tau \leq c, \\ e^{-\gamma(\tau - c)} & , \tau > c, \end{cases} \tag{9}$$

and

$$\sigma(\tau) = \begin{cases} 0 & ,0 \le \tau \le c, \\ \bar{\sigma} & ,\tau > c. \end{cases} \tag{10}$$

We interpret c as the length of latent (exposed) period where an individual is infected but is still not capable of spreading the diseases. After the latent period, the infective individual spreads the disease with the (mass-action) rate $\bar{\sigma}$, and the probability that the individual still stays infective is exponentially distributed with the parameter $-\gamma$.

(a) Calculate the expected amount of time that an infective individual spends in the infective class, and also the expected amount of time that an infective individual is capable of spreading the disease.

(b) Calculate the explicit expressions for κ and y_0 in (4), simplify the expressions as much as you can. What is this model's $\mathcal{R}_0$?

(c) Define a function $E : [0, \infty) \to [0, \infty)$ by

$$E(t) = \int_0^c i(t, \tau)\, \mathrm{d}\tau, \quad t \ge 0. \tag{11}$$

The function $E(t)$ gives the density of individuals at time t that are infected but are still not capable of spreading the diseases (the "exposed" class).

Define a function $I : [0, \infty) \to [0, \infty)$ by

$$I(t) = \int_c^\infty i(t, \tau)\, \mathrm{d}\tau, \quad t \ge 0, \tag{12}$$

$$\tag{13}$$

The function $I(t)$, for $t \ge 0$, gives the density of individuals at time t that are infected and are capable of spreading the diseases.

Define $R : [0, \infty) \to [0, \infty)$ to be the solution of the initial value problem:

$$\dot{R}(t) = -\int_c^\infty i(t, \tau) \frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\, \mathrm{d}\tau, \quad R(0) = 0. \tag{14}$$

The function $R(t)$ gives the density of individuals at time t that are recovered or removed.

Assume that the function E and I are continuous. Assume that the function $t \mapsto \int_c^\infty i(t, \tau) \frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\, \mathrm{d}\tau$ is continuous, and so the function R is well-defined and is continuously differentiable (in fact, $R(t) = -\int_0^t \int_c^\infty i(s, \tau) \frac{\mathcal{F}'(\tau)}{\mathcal{F}(\tau)}\, \mathrm{d}\tau\, \mathrm{d}s$).

Show that, for $t \ge 0$, the solution of (1)–(3) satisfies the system of delay differential equations:

$$\dot{S}(t) = -\bar{\sigma} S(t) I(t), \tag{15}$$

$$\dot{E}(t) = \begin{cases} \bar{\sigma} S(t) I(t) - \bar{\sigma} S(t-c) I(t-c) & ,t > c, \\ \bar{\sigma} S(t) I(t) - i_0(c-t) & ,t < c, \end{cases} \tag{16}$$

$$\dot{I}(t) = \begin{cases} \bar{\sigma} S(t-c) I(t-c) - \gamma I(t) & ,t > c \\ -\gamma I(t) + +i_0(c-t) & ,t < c, \end{cases} \tag{17}$$

$$\dot{R}(t) = \gamma I(t). \tag{18}$$

Hint: It should not be hard to verify (15) and (18). Verifying (16) and (17) takes more work, and you should consider the cases $t \ge c$ and $0 \le t < c$ separately.