Note: you do not need to follow the hints.

1. Consider the simplified Kermark-McKendrick model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sigma SI, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \sigma SI - \gamma I, \tag{1}$$

with $S(0) > 0, I(0) > 0$, $\sigma > 0$, $\gamma > 0$.

Define $\mathcal{R}(S) = \frac{\sigma S}{\gamma}$ and note that $\mathcal{R}_0 = \mathcal{R}(S(0))$ is the basic reproductive number of the model.

Suppose that $\mathcal{R}_0 > 1$, i.e., there is an outbreak of the epidemic. Show that the density of infectives reaches the maximum value exactly at time t' where $t' > 0$ is the unique real number such that $\mathcal{R}(S(t')) = 1$. Can you see why this should make biological sense? Write out the explicit formula for the maximum density of infectives.

Hint: Recall that the function $V(S, I) = S + I - \frac{\gamma}{\sigma} \log S$ is constant along the solution of (1).

2. [an SI epidemic model] Consider the simplified Kermark-McKendrick model, as above. If we assume that $\gamma = 0$, so no individual can move out of the I class. Then once an individual is in the I class (is infected), it will remain capable of spreading the disease for all future time. The reduced model is then:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= -\sigma SI, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma SI, \end{aligned} \tag{2}$$

with $S(0) \geq 0$, $I(0) \geq 0$, $\sigma > 0$. Show that:

(a) There is a unique non-negative solution of (2) defined on $t \in [0, \infty)$.

(b) (1) If $I(0) = 0$, then $I(t) = 0$ and $S(t) = S(0)$ for all $t \in [0, \infty)$.
 (2) If $S(0) = 0$, then $I(t) = I(0)$ and $S(t) = 0$ for all $t \in [0, \infty)$.
 (3) If $S(0) > 0$ and $I(0) > 0$, then $S(t)$ decreases monotonically toward 0 as $t \to \infty$ and $I(t)$ increases monotonically toward $S(0) + I(0)$ as $t \to \infty$. This means that every individual in the population eventually becomes infective.

(c) Do you think this is a good model? Why or why not?

3. Recall the equation for the final size, S_∞, of the simplified Kermark-McKendrick model,

$$S_\infty - \frac{\gamma}{\sigma} \log S_\infty = S(0) + I(0) - \frac{\gamma}{\sigma} \log S(0). \tag{3}$$

Suppose $S(0), I(0), \sigma, \gamma > 0$ are given. Show that there are two values of $S_\infty > 0$ that satisfies (3), and that the final size is the smaller of the two.

Hint: Consider the shape of the graph of $f(x) = x - \frac{\gamma}{\sigma} \log x$, $x > 0$. Pay attention to the value of x where $f(x)$ has minimum value.

4. There was an outbreak of an infectious disease in a city of 1 million inhabitants. The average length of infectious period of the disease was 7 days. At the end of the epidemic, there were 70,000 people that didn't contact the disease. Assume that the outbreak can be describe by the simplified Kermark-McKendrick model, and that the initial number of infectives was so small that it can be neglected. Find the basic reproductive number, $\mathcal{R}_0$, for this outbreak.