1. Show directly from the definition that the following functions are locally Lipschitz continuous (with respect to the second argument).

(a) $f(t,x) = t\cos(x) - \sin(x)$.

Let $(t_0, x_0) \in (-\infty, \infty) \times \mathbb{R}$. Let ε be any positive number, and let $\Lambda = \max(|t_0 - \varepsilon|, |t_0 + \varepsilon|) + 1$. Then for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $x, x' \in \mathbb{R}$ such that $|x - x_0|, |x' - x_0| < \varepsilon$, we have

$$\begin{aligned} |f(t,x) - f(t,x')| &= |t\cos(x) - \sin(x) - t\cos(x') + \sin(x')| \\ &\le |t| \cdot |\cos(x) - \cos(x')| + |\sin(x) - \sin(x')|. \end{aligned}$$

By the mean value theorem, there exist c_1 and c_2 such that $\cos(x) - \cos(x') = -\sin(c_1)(x - x')$ and $\sin(x) - \sin(x') = \cos(c_2)(x - x')$, and so we have

$$\begin{aligned} |f(t,x) - f(t,x')| &\le |t| \cdot |\cos(x) - \cos(x')| + |\sin(x) - \sin(x')| \\ &= |t| \cdot |\sin(c_1)| \cdot |x - x'| + |\cos(c_2)| \cdot |x - x'| \\ &\le (|t| + 1) \cdot |x - x'| \\ &\le \Lambda |x - x'|. \end{aligned}$$

(b) $f(t,x) = t^2|x+1|$.

Let $(t_0, x_0) \in (-\infty, \infty) \times \mathbb{R}$. Let ε be any positive number, and let $\Lambda = \max((t_0 - \varepsilon)^2, (t_0 + \varepsilon)^2)$. Then for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $x, x' \in \mathbb{R}$ such that $|x - x_0|, |x' - x_0| < \varepsilon$, we have

$$\begin{aligned} |f(t,x) - f(t,x')| &= \big|t^2|x+1| - t^2|x'+1|\big| \\ &= t^2\big||x+1| - |x'+1|\big| \\ &\le t^2|x + 1 - x' - 1| \\ &\le \Lambda |x - x'|. \end{aligned}$$

2.

(a) Show that the function $f(t,x) = (x-1)^{1/3}$, $(t,x) \in (-\infty, \infty) \times \mathbb{R}$, is not locally Lipschitz continuous (with respect to the second argument).

From the definition of locally Lipschitz continuity, a function is *not* locally Lipschitz continuous if (and only if) there exists $(t_0, x_0) \in (-\infty, \infty) \times \mathbb{R}$ such that for every $\varepsilon > 0$ and any $\Lambda > 0$, there exist $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $x, x' \in \mathbb{R}$ with $|x - x_0|, |x' - x_0| < \varepsilon$ and $|f(t,x) - f(t,x')| > \Lambda |x - x'|$.

Let $(t_0, x_0) = (0, 1)$ and let $t = t_0 = 0$, $x = x_0 = 1$. For any $\varepsilon > 0$ and any $\Lambda > 0$, we let $a = \min(1, \varepsilon^{2/3})$ and let $x' = 1 + (a/(\Lambda+1))^{3/2}$. Then $|x' - x_0| = |x' - 1| = (a/(\Lambda+1))^{3/2} < a^{3/2} \le \varepsilon$ and

$$\begin{aligned} |f(t,x') - f(t,x)| &= |(x'-1)^{1/3}| \\ &= |(x'-1)^{-2/3}| \cdot |x' - 1| \\ &= \big(a/(\Lambda+1)\big)^{-1} \cdot |x' - x| \\ &= \big((\Lambda+1)/a\big) \cdot |x' - x| \\ &> \Lambda |x' - x| \end{aligned}$$

(b) Find at least 3 solutions of the initial value problem $\dot{x} = (x-1)^{1/3}, x(0) = 1$.

For each $b \in \{-1, 1\}$ and $c \geq 0$, define $y_{bc} : \mathbb{R} \to \mathbb{R}$ as

$$y_{bc}(t) = \begin{cases} 1 & , t < c, \\ b\left(\frac{2}{3}(t-c)\right)^{3/2} + 1 & , t \geq c. \end{cases}$$

Then y_{bc} is a solution of $\dot{x} = (x-1)^{1/3}, x(0) = 1$. Also $y(t) = 1$, $t \in \mathbb{R}$, is a solution of $\dot{x} = (x-1)^{1/3}, x(0) = 1$.

3.

(a) Let $I = (-\infty, \infty)$ and $D = (0, \infty)$. Define a function $f : I \times D \to \mathbb{R}$ by $f(t, x) = x^{m+1}$ where m is a positive real number. Find a solution to the initial value problem $\dot{x} = f(t, x), x(0) = a$, $a \in D$. Determine the maximal interval on which the solution is defined. Describe how the solution behaves at the endpoint of the interval and how the endpoint depends on m and a.

The solution is $x(t) = \frac{1}{(a^{-m}-mt)^{1/m}}$, which is defined on $t \in (-\infty, \frac{1}{ma^m})$. We note that this is the only solution since f is locally Lipschitz continuous on $I \times D$. We have $x(t) \nearrow \infty$ as $t \nearrow \frac{1}{ma^m}$. The "blow up time" is a decreasing function of both m and a.

(b) Let $I = (-\infty, \infty)$ and $D = (0, \infty)$. Define a function $f : I \times D \to \mathbb{R}$ by $f(t, x) = -x^{1-m}$ where m is an even positive integer. Find a solution to the initial value problem $\dot{x} = f(t, x), x(0) = a$, $a \in D$. Determine the maximal interval on which the solution is defined. Describe how the solution behaves at the endpoint of the interval and how the endpoint depends on m and a.

The solution is $x(t) = (a^m - mt)^{1/m}$, which, for even positive integer m, is defined on $t \in (-\infty, \frac{a^m}{m})$. Again, this is the only solution since f is locally Lipschitz continuous on $I \times D$. As $t \nearrow \frac{a^m}{m}$, we have $x(t) \searrow 0$, and 0 is not in D, i.e., the solution "slips out" of the domain of the vector field.

Incidentally, we do not really require that m is an even positive integer, any real number $m > 1$ will give us the same result, i.e., $x(t) \searrow 0$ as $t \nearrow \frac{a^m}{m}$. Notice that if $m = 3$, the expression $x(t) = (a^m - mt)^{1/m}$ is defined for all $t \in (-\infty, \infty)$, but is only a solution of the initial value problem on $t \in (-\infty, \frac{a^m}{m})$!.

4. [Simple SIRS model] Consider the following initial value problem:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= -\sigma SI + \beta R, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma SI - \gamma I, \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - \beta R, \end{aligned}$$

with $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$. Here, β is the per-capita rate of losing immunity. Assume that the parameters σ, γ, and β are all strictly positive. Show that:

(a) There is a solution defined on $t \in [0, b)$ for some $0 < b \leq \infty$.

Here, the vector field is $f : (-\infty, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(t, S, I, R) = \begin{pmatrix} -\sigma SI + \beta R \\ \sigma SI - \gamma I \\ \gamma I - \beta R. \end{pmatrix}$$

It is easy to see that f is continuous, and so there exists a solution $S(t), I(t), R(t)$ defined on some open interval containing 0.

(b) There can be only one solution on $t \in [0, b)$.

Since the vector field f is continuously differentiable with respect to $(S, I, R) \in \mathbb{R}^3$, we know that f is locally Lipschitz continuous, and so the initial value problem has a unique solution.

(c) If $S(t), I(t), R(t)$ is a solution defined on $t \in [0, b)$, then $S(t) \geq 0$, $I(t) \geq 0$, and $R(t) \geq 0$ for all $t \in [0, b)$, and $S(t) + I(t) + R(t)$ is constant on $[0, b)$.

We know that $S(t) + I(t) + R(t)$ is constant since $\frac{\mathrm{d}}{\mathrm{d}t}(S(t) + I(t) + R(t)) = 0$. From the equation $\frac{\mathrm{d}I}{\mathrm{d}t}(t) = (\sigma S(t) - \gamma)I(t)$, $I(0) \geq 0$, we have $I(t) = I(0)e^{\int_0^t (\sigma S(\tau) - \gamma)\,\mathrm{d}\tau} \geq 0$. Since $I(t) \geq 0$, we have $\frac{\mathrm{d}R}{\mathrm{d}t}(t) = \gamma I(t) - \beta R(t) \geq -\beta R(t)$, $R(0) \geq 0$, and so $R(t) \geq R(0)e^{-\beta t} \geq 0$. Since $R(t) \geq 0$, we have $\frac{\mathrm{d}S}{\mathrm{d}t}(t) = -\sigma S(t)I(t) + \beta R(t) \geq -\sigma S(t)I(t)$, $S(0) \geq 0$, and so $S(t) \geq S(0)e^{-\sigma \int_0^t I(\tau)\,\mathrm{d}\tau} \geq 0$.

(d) There exists a unique non-negative solution defined for all non-negative time, i.e., we can have $b = \infty$.

Suppose that we cannot have $b = \infty$. Let $b < \infty$ be such that the solution on $[0, b)$ cannot be extended further to the right, i.e., there is not solution on $[0, c)$ for any $c > b$. Then, since the vector field is defined on $(-\infty, \infty) \times \mathbb{R}^3$, the solution cannot "slip out" of the domain of the vector field. Hence, we must have $\limsup_{t \nearrow b}(|S(t)| + |I(t)| + |R(t)|) = \infty$. But since $0 \leq S(t) + I(t) + R(t) \leq S(0) + I(0) + R(0)$ for $t \in [0, b)$ by part (c), this is impossible.