Note: you do not need to follow the hints.

1. Show directly from the definition that the following functions are locally Lipschitz continuous (with respect to the second argument).

(a) $f(t,x) = t\cos(x) - \sin(x)$.

(b) $f(t,x) = t^2|x+1|$.

Hint: For (a), the mean value theorem from calculus should be useful. For (b), the so-called reverse triangle inequality: $\big||a| - |b|\big| \le |a-b|$, for $a, b \in \mathbb{R}$, should be useful.

2.

(a) Show that the function $f(t,x) = (x-1)^{1/3}$, $(t,x) \in (-\infty,\infty) \times \mathbb{R}$, is not locally Lipschitz continuous (with respect to the second argument).

Hint:

(1) Show that $|f(t,x) - f(t,1)| = |(x-1)^{-2/3}| \cdot |x-1|$ for $x \neq 1$, $t \in (-\infty,\infty)$.

(2) Show that for any $\varepsilon > 0$, there is no finite $\Lambda > 0$ such that $|f(t,x) - f(t,x')| \le \Lambda|x-x'|$ for all $t \in (-\varepsilon, +\varepsilon)$ and all $x, x' \in (1-\varepsilon, 1+\varepsilon)$.

(3) Explain why this implies that f is not locally Lipschitz continuous.

(b) Find at least 3 solutions of the initial value problem $\dot{x} = (x-1)^{1/3}, x(0) = 1$.

3.

(a) Let $I = (-\infty,\infty)$ and $D = (0,\infty)$. Define a function $f : I \times D \to \mathbb{R}$ by $f(t,x) = x^{m+1}$ where m is a positive real number. Find a solution to the initial value problem $\dot{x} = f(t,x), x(0) = a$, $a \in D$. Determine the maximal interval on which the solution is defined. Describe how the solution behaves at the endpoint of the interval and how the endpoint depends on m and a.

(b) Let $I = (-\infty,\infty)$ and $D = (0,\infty)$. Define a function $f : I \times D \to \mathbb{R}$ by $f(t,x) = -x^{1-m}$ where m is an even positive integer. Find a solution to the initial value problem $\dot{x} = f(t,x), x(0) = a$, $a \in D$. Determine the maximal interval on which the solution is defined. Describe how the solution behaves at the endpoint of the interval and how the endpoint depends on m and a.

4. [Simple SIRS model] Consider the following initial value problem:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sigma SI + \beta R,$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \sigma SI - \gamma I,$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \beta R,$$

with $S(0) \ge 0$, $I(0) \ge 0$, $R(0) \ge 0$. Here, β is the per-capita rate of losing immunity. Assume that the parameters σ, γ, and β are all strictly positive. Show that:

(a) There is a solution defined on $t \in [0,b)$ for some $0 < b \le \infty$.

(b) There can be only one solution on $t \in [0,b)$.

(c) If $S(t), I(t), R(t)$ is a solution defined on $t \in [0, b)$, then $S(t) \geq 0$, $I(t) \geq 0$, and $R(t) \geq 0$ for all $t \in [0, b)$, and $S(t) + I(t) + R(t)$ is constant on $[0, b)$.

Hint: First show that $I(t) \geq 0$. Then show that $R(t) \geq 0$ because $\frac{\mathrm{d}R}{\mathrm{d}t} \geq -\beta R$. Then show that $S(t) \geq 0$ because $\frac{\mathrm{d}S}{\mathrm{d}t} \geq -\sigma S I$.

(d) There exists a unique non-negative solution defined for all non-negative time, i.e., we can have $b = \infty$.